

On a higher level extension of Leclerc-Thibon product theorem in q -deformed Fock spaces

Kazuto Iijima

Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan

kazuto.iijima@math.nagoya-u.ac.jp

Abstract

The q -deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado. Uglov defined a canonical bases in q -deformed Fock spaces of higher levels. Leclerc-Thibon showed a product theorem in q -deformed Fock spaces of level one. The product theorem is regarded as a formal q -analogue of the tensor product theorem of level one. In this paper, we show a higher level analogue of Leclerc-Thibon product theorem under a suitable multi charge condition.

1 Introduction

The q -deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado [JMMO91]. For integers $n \geq 2$, $\ell \geq 1$ and a multi charge $s = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$, the q -deformed Fock space $F_q[s]$ of level ℓ is the $\mathbb{Q}(q)$ -vector space whose basis are indexed by ℓ -tuples of Young diagrams. i.e. $\{|\lambda; s\rangle \mid \lambda \in \Pi^\ell\}$, where Π is the set of Young diagrams.

The Fock space $F_q[s]$ is endowed with the action of bosons B_m and they generate a Heisenberg algebra [Ugl00]. For a partition λ , we define S_λ as a \mathbb{Q} -linear combination of products of elements B_{-m} . (See §3.1 for the precise definition.) Quantum group $U_q(\hat{\mathfrak{sl}}_n)$ also acts on $F_q[s]$ as level q^ℓ . These actions commute on $F_q[s]$.

The canonical bases $\{G^+(\lambda; s) \mid \lambda \in \Pi^\ell\}$ and $\{G^-(\lambda; s) \mid \lambda \in \Pi^\ell\}$ are bases of the Fock space $F_q[s]$ that are invariant under a certain involution $\bar{}$ [Ugl00].

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is n -restricted if $0 \leq \lambda_i - \lambda_{i+1} \leq n$ for all $i = 1, 2, \dots$. Each partition λ can be uniquely written as $\lambda = \tilde{\lambda} + n\check{\lambda}$, where $\tilde{\lambda}, \check{\lambda} \in \Pi$ and $\tilde{\lambda}$ is n -restricted. In the case of $\ell = 1$, Leclerc and Thibon introduced linear operators S_λ as a \mathbb{Q} -linear combination of products of elements B_{-m} and showed the following theorem (See §3.1 for the precise definitions) :

Theorem A. (Leclerc-Thibon[LT00])

Suppose that $\ell = 1$. Let $\lambda = \tilde{\lambda} + n\check{\lambda}$. Then,

$$G^-(\lambda) = S_{\check{\lambda}} G^-(\tilde{\lambda}).$$

Our main result is a higher level version of Theorem A. For a non-negative integer M , a basis vector $|\lambda; s\rangle$ is called M -dominant if $s_i - s_{i+1} \geq M + |\lambda|$ for all $i = 1, 2, \dots, \ell - 1$. For $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)}) \in \Pi^\ell$, define $\widetilde{\lambda}$ and $\check{\lambda}$ as $\widetilde{\lambda} = (\widetilde{\lambda}^{(1)}, \widetilde{\lambda}^{(2)}, \dots, \widetilde{\lambda}^{(\ell)})$, $\check{\lambda} = (\check{\lambda}^{(1)}, \check{\lambda}^{(2)}, \dots, \check{\lambda}^{(\ell)})$.

Theorem B. (Theorem 4.12)

If $|\lambda; s\rangle$ is 0-dominant, then

$$G^-(\lambda; s) = S_{\check{\lambda}} G^-(\widetilde{\lambda}; s),$$

where $S_{\check{\lambda}}$ is a linear operator on $F_q[s]$ defined in Definition 4.11.

Now we explain a expected connections of canonical bases $G^\pm(\lambda; s)$ with representation theory. Define matrices $\Delta^+(q) = (\Delta_{\lambda, \mu}^+(q))_{\lambda, \mu}$ and $\Delta^-(q) = (\Delta_{\lambda, \mu}^-(q))_{\lambda, \mu}$ by $G^+(\lambda; s) = \sum_{\mu} \Delta_{\lambda, \mu}^+(q) |\mu; s\rangle$, $G^-(\lambda; s) = \sum_{\mu} \Delta_{\lambda, \mu}^-(q) |\mu; s\rangle$. We call $\Delta_{\lambda, \mu}^+(q)$ and $\Delta_{\lambda, \mu}^-(q)$ q -decomposition numbers. These q -decomposition matrices play an important role in representation theory. However it is not known that there is a explicit combinatorial formula which expresses q -decomposition matrices as an one dimensional summation.

In the case of $\ell = 1$, Varagnolo-Vasserot [VV99] proved that $\Delta^+(1)$ coincides with the decomposition matrix of v -Schur algebra. Ariki defined a q -analogue of decomposition numbers of v -Schur algebra by using Khovanov-Lauda's grading, and proved that it coincides with the q -decomposition numbers [Ari].

We recall the tensor product theorem by Lusztig [Lus89]. Let $\zeta \in \mathbb{C}$ be such that ζ^2 is a primitive n -th root of unity. Let Fr denote the Frobenius map from the quantum enveloping algebra $U_\zeta(\mathfrak{gl}_r)$ to the classical enveloping algebra $U(\mathfrak{gl}_r)$. Given a $U(\mathfrak{gl}_r)$ -module M , one can define a $U_\zeta(\mathfrak{gl}_r)$ -module M^{Fr} by composing the action of $U(\mathfrak{gl}_r)$ with Fr .

Theorem (Steinberg-Lusztig [Lus89])

Suppose that $\ell = 1$. Let $\lambda = \widetilde{\lambda} + n\check{\lambda}$. Then,

$$L(\lambda) = L(\widetilde{\lambda}) \otimes W(\check{\lambda})^{\text{Fr}},$$

where $L(\lambda)$ (resp. $W(\mu)$) is the simple (resp. classical Weyl) module with highest weight λ (μ).

Since the simple module $L(\lambda)$ corresponds to the canonical basis $G^-(\lambda)$ in the case of $\ell = 1$, Theorem A is a Fock space version of Steinberg-Lusztig's tensor product theorem. (see [LT00] for details.)

For $\ell \geq 2$, Yvonne [Yvo06] conjectured that the matrix $\Delta^+(q)$ coincides with the q -analogue of the decomposition matrices of cyclotomic Schur algebras at a primitive n -th root of unity under a suitable condition on a multi charge and a proof of the conjecture is presented by Stroppel-Webster [SW, Theorem D].

Let $\mathcal{O}_s(\ell, 1, m)$ be the category \mathcal{O} of rational Cherednik algebra of $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_m$ associated with multi charge s [GGOR03]. Rouquier [Rou08, Theorem 6.8, §6.5] conjectured that, for an arbitrary multi charge, the multiplicities of simple modules in standard modules in $\mathcal{O}_s(\ell, 1, m)$ are equal to the corresponding coefficients $\Delta_{\lambda, \mu}^+(q)$, where $m = |\lambda| = |\mu|$. Shan showed that $\oplus_{m \geq 0} \mathcal{O}_s(\ell, 1, m)$ categorify $F_1[s]$ [Sha]. More generally, it is expected that, together with a suitable grading, $\oplus_{m \geq 0} \mathcal{O}_s(\ell, 1, m)$ should categorify $F_q[s]$. For the detail of correspondence between the charges of $\mathcal{O}_s(\ell, 1, m)$ and the charges of Fock spaces, see [Rou08].

This paper is organized as follows. In Section 2, we review the q -deformed Fock spaces of higher levels and their canonical bases. In Section 3, we review the q -analogue of tensor product theorem by Leclerc-Thibon. In Section 4, we state the main results and prove them except Proposition 4.6. In Section 5, we prove Proposition 4.6.

Acknowledgments

I am deeply grateful to Hyohe Miyachi, Soichi Okada, Toshiaki Shoji and Kentaro Wada for their advice.

Notations

For a positive integer N , a *partition* of N is a non-increasing sequence of non-negative integers summing to N . We write $|\lambda| = N$ if λ is a partition of N . The *length* $l(\lambda)$ of λ is the number of non-zero components of λ . And we use the same notation λ to represent the Young diagram corresponding to λ . For an ℓ -tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)})$ of Young diagrams, we put $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| + \dots + |\lambda^{(\ell)}|$.

2 The q -deformed Fock spaces of higher levels

2.1 q -wedge products and straightening rules

Let n, ℓ, s be integers such that $n \geq 2$ and $\ell \geq 1$. Let $r \in \mathbb{Z}_{\geq 0}$. For $\mathbf{k} \in \mathbb{Z}^r$, a *finite q -wedge of length r* is

$$u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r}.$$

Finite q -wedges satisfy certain commutation relations, so-called *straightening rules*. Note that the straightening rules depend on n and ℓ . [Ugl00, Proposition 3.16]

Example 2.1. (i) For every $k_1 \in \mathbb{Z}$, $u_{k_1} \wedge u_{k_1} = -u_{k_1} \wedge u_{k_1}$. Therefore $u_{k_1} \wedge u_{k_1} = 0$.

(ii) Let $n = 2$, $\ell = 2$, $k_1 = -2$, and $k_2 = 4$. Then

$$u_{-2} \wedge u_4 = q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0.$$

(iii) Let $n = 2$, $\ell = 2$, $k_1 = -1$, $k_2 = -2$ and $k_3 = 4$. Then,

$$\begin{aligned} u_{-1} \wedge u_{-2} \wedge u_4 &= u_{-1} \wedge (u_{-2} \wedge u_4) = u_{-1} \wedge (q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0) \\ &= q u_{-1} \wedge u_4 \wedge u_{-2} + (q^2 - 1) u_{-1} \wedge u_2 \wedge u_0 \\ &= -u_4 \wedge u_{-1} \wedge u_{-2} - (q - q^{-1}) u_3 \wedge u_0 \wedge u_{-2} - (q - q^{-1}) u_2 \wedge u_0 \wedge u_{-1}. \end{aligned}$$

We define $P(s)$ and $P^{++}(s)$ as follows;

$$P(s) = \{\mathbf{k} = (k_1, k_2, \dots) \in \mathbb{Z}^\infty \mid k_r = s - r + 1 \text{ for any sufficiently large } r\} \quad (1)$$

$$P^{++}(s) = \{\mathbf{k} = (k_1, k_2, \dots) \in P(s) \mid k_1 > k_2 > \dots\}. \quad (2)$$

Let Λ^s be the $\mathbb{Q}(q)$ vector space spanned by the q -wedge products

$$u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \dots, \quad (\mathbf{k} \in P(s)) \quad (3)$$

subject to straightening rules. We regard a finite q -wedge product $u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r}$ as the infinite q -wedge product

$$u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r} \wedge u_{s-r} \wedge u_{s-r-1} \wedge u_{s-r-2} \wedge \dots. \quad (4)$$

By applying the straightening rules, every q -wedge product $u_{\mathbf{k}}$ is expressed as a linear combination of so-called *ordered q -wedge products*, namely q -wedge products $u_{\mathbf{k}}$ with $\mathbf{k} \in P^{++}(s)$. The ordered q -wedge products $\{u_{\mathbf{k}} \mid \mathbf{k} \in P^{++}(s)\}$ form a basis of Λ^s called *the standard basis*.

2.2 Abacus

It is convenient to use the abacus notation for studying various properties in straightening rules.

Fix an integer $N \geq 2$, and form an infinite abacus with N runners labeled $1, 2, \dots, N$ from left to right. The positions on the i -th runner are labeled by the integers having residue i modulo N .

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ -N+1 & -N+2 & \dots & -1 & 0 \\ 1 & 2 & \dots & N-1 & N \\ N+1 & N+2 & \dots & 2N-1 & 2N \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Each $\mathbf{k} \in P^{++}(s)$ (or the corresponding q -wedge product $u_{\mathbf{k}}$) can be represented by a bead-configuration on the abacus with $n\ell$ runners and beads put on the positions k_1, k_2, \dots . We call this configuration *the abacus presentation* of $u_{\mathbf{k}}$.

Example 2.2. If $n = 2$, $\ell = 3$, $s = 0$, and $\mathbf{k} = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \dots)$, then the abacus presentation of $u_{\mathbf{k}}$ is

$d = 1$		$d = 2$		$d = 3$		
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$\textcircled{-17}$	$\textcircled{-16}$	$\textcircled{-15}$	$\textcircled{-14}$	$\textcircled{-13}$	$\textcircled{-12}$	$\cdots m = 3$
$\textcircled{-11}$	$\textcircled{-10}$	$\textcircled{-9}$	$\textcircled{-8}$	$\textcircled{-7}$	-6	$\cdots m = 2$
$\textcircled{-5}$	$\textcircled{-4}$	-3	$\textcircled{-2}$	-1	0	$\cdots m = 1$
$\textcircled{1}$	$\textcircled{2}$	$\textcircled{3}$	4	5	$\textcircled{6}$	$\cdots m = 0$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$c = 1$	$c = 2$	$c = 1$	$c = 2$	$c = 1$	$c = 2$	

We use another labeling of runners and positions. Given an integer k , let c, d and m be the unique integers satisfying

$$k = c + n(d - 1) - n\ell m \quad , \quad 1 \leq c \leq n \quad \text{and} \quad 1 \leq d \leq \ell. \quad (5)$$

Then, in the abacus presentation, the position k is on the $c + n(d - 1)$ -th runner (see the previous example). Relabeling the position k by $c - nm$, we have ℓ abaci with n runners.

Example 2.3. In the previous example, relabeling the position k by $c - nm$, we have

$d = 1$		$d = 2$		$d = 3$		
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$\textcircled{-5}$	$\textcircled{-4}$	$\textcircled{-5}$	$\textcircled{-4}$	$\textcircled{-5}$	$\textcircled{-4}$	$\cdots m = 3$
$\textcircled{-3}$	$\textcircled{-2}$	$\textcircled{-3}$	$\textcircled{-2}$	$\textcircled{-3}$	-2	$\cdots m = 2$
$\textcircled{-1}$	$\textcircled{0}$	-1	$\textcircled{0}$	-1	0	$\cdots m = 1$
$\textcircled{1}$	$\textcircled{2}$	$\textcircled{1}$	2	1	$\textcircled{2}$	$\cdots m = 0$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$c = 1$	$c = 2$	$c = 1$	$c = 2$	$c = 1$	$c = 2$	

We assign to each of ℓ abacus presentations with n runners a q -wedge product of level 1. In fact, straightening rules in each “sector” are the same as those of level 1 by identifying the abacus in the sector with that of level 1. (see also [Ugl00] and [Iij12])

We introduce some notation.

Definition 2.4. For an integer k , let c, d and m be the unique integers satisfying (5), and write

$$u_k = u_{c-nm}^{(d)}. \quad (6)$$

Also we write $u_{c_1-nm_1}^{(d_1)} > u_{c_2-nm_2}^{(d_2)}$ if $k_1 > k_2$, where $k_i = c_i + n(d_i - 1) - n\ell m_i$, ($i = 1, 2$).

A finite q -wedge

$$v = u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_2)} \wedge \cdots \wedge u_{k_r}^{(d_r)}$$

is simple if $d_1 = d_2 = \cdots = d_r$.

We regard $u_{c-nm}^{(d)}$ as u_{c-nm} in the case of $\ell = 1$.

Example 2.5. If $n = 2$, $\ell = 3$, then we have

$$u_{-10} \wedge u_1 = -q^{-1} u_1 \wedge u_{-10} + (q^{-2} - 1) u_{-4} \wedge u_{-5},$$

that is,

$$u_{-2}^{(1)} \wedge u_1^{(1)} = -q^{-1} u_1^{(1)} \wedge u_{-2}^{(1)} + (q^{-2} - 1) u_0^{(1)} \wedge u_{-1}^{(1)}.$$

On the other hand, in the case of $n = 2$, $\ell = 1$,

$$u_{-2} \wedge u_1 = -q^{-1} u_1 \wedge u_{-2} + (q^{-2} - 1) u_0 \wedge u_{-1}.$$

2.3 ℓ -tuples of Young diagrams

Another indexation of the ordered q -wedge products is given by the set of pairs (λ, s) of ℓ -tuples of Young diagrams $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and integer sequence $s = (s_1, \dots, s_\ell)$ summing up to s . Let $\mathbf{k} = (k_1, k_2, \dots) \in P^{++}(s)$, and write

$$k_r = c_r + n(d_r - 1) - n\ell m_r, \quad 1 \leq c_r \leq n, \quad 1 \leq d_r \leq \ell, \quad m_r \in \mathbb{Z}.$$

For $d \in \{1, 2, \dots, \ell\}$, let $k_1^{(d)}, k_2^{(d)}, \dots$ be integers such that

$$\beta^{(d)} = \{c_r - nm_r \mid d_r = d\} = \{k_1^{(d)}, k_2^{(d)}, \dots\} \quad \text{and} \quad k_1^{(d)} > k_2^{(d)} > \cdots$$

Then we associate to the sequence $(k_1^{(d)}, k_2^{(d)}, \dots)$ an integer s_d and a partition $\lambda^{(d)}$ by

$$k_r^{(d)} = s_d - r + 1 \quad \text{for sufficiently large } r \quad \text{and} \quad \lambda_r^{(d)} = k_r^{(d)} - s_d + r - 1 \quad \text{for } r \geq 1.$$

In this correspondence, we also write

$$u_{\mathbf{k}} = |\lambda; s\rangle \quad (\mathbf{k} \in P^{++}(s)). \tag{7}$$

Example 2.6. If $n = 2$, $\ell = 3$, $s = 0$, and $\mathbf{k} = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \dots)$, then

$$\begin{aligned} k_1 &= 6 = 2 + 2(3 - 1) - 6 \cdot 0, & k_2 &= 3 = 1 + 2(2 - 1) - 6 \cdot 0, \\ k_3 &= 2 = 2 + 2(1 - 1) - 6 \cdot 0, & \dots & \text{and so on.} \end{aligned}$$

Hence,

$$\beta^{(1)} = \{2, 1, 0, -1, -2, \dots\}, \quad \beta^{(2)} = \{1, 0, -2, -3, -4, \dots\}, \quad \beta^{(3)} = \{2, -3, -4, -5, \dots\}.$$

Thus, $s = (2, 0, -2)$ and $\lambda = (\emptyset, (1, 1), (4))$.

Note that we can read off $s = (2, 0, -2)$ and $\lambda = (\emptyset, (1, 1), (4))$ from the abacus presentation. (see Example 2.3)

2.4 The q -deformed Fock spaces of higher levels

Definition 2.7. For $s \in \mathbb{Z}^\ell$, we define the q -deformed Fock space $F_q[s]$ of level ℓ to be the subspace of Λ^s spanned by $|\lambda; s\rangle$ ($\lambda \in \Pi^\ell$):

$$F_q[s] = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{Q}(q) |\lambda; s\rangle. \quad (8)$$

We call s a multi charge.

2.5 The action of bosons

The Fock space $F_q[s]$ is endowed with the action of bosons B_m given by

$$B_m(u_k) = \sum_{r \geq 1} u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_{r-1}} \wedge u_{k_r - n\ell m} \wedge u_{k_{r+1}} \wedge \cdots, \quad (m \in \mathbb{Z}^*), \quad (9)$$

where \mathbb{Z}^* denotes the set of nonzero integers.

Remark. B_m 's generate a Heisenberg algebra [Ugl00, Proposition 4.5]:

$$[B_m, B_{m'}] = \delta_{m, -m'} m \frac{1 - q^{-2mn}}{1 - q^{-2m}} \cdot \frac{1 - q^{2m\ell}}{1 - q^{2m}}, \quad (m \in \mathbb{Z}_{>0} \text{ and } m' \in \mathbb{Z}^*).$$

2.6 The bar involution

Definition 2.8. The bar involution $\overline{}$ of Λ^s is the \mathbb{Q} -vector space automorphism such that $\overline{q} = q^{-1}$ and

$$\overline{u_k} = \overline{u_{k_1} \wedge \cdots \wedge u_{k_r} \wedge u_{k_{r+1}} \wedge \cdots} = (-q)^{\kappa(d_1, \dots, d_r)} q^{-\kappa(c_1, \dots, c_r)} (u_{k_r} \wedge \cdots \wedge u_{k_1}) \wedge u_{k_{r+1}} \wedge \cdots, \quad (10)$$

where c_i, d_i are defined by k_i as in (5), r is an integer satisfying $k_r = s - r + 1$. And $\kappa(a_1, \dots, a_r)$ is defined by

$$\kappa(a_1, \dots, a_r) = \#\{(i, j) \mid i < j, a_i = a_j\}.$$

Remarks.

1. The involution is well defined. i.e. it doesn't depend on the choice of r [Ugl00].
2. The involution comes from the bar involution of affine Hecke algebra \hat{H}_r . (see [Ugl00] for more detail.)
3. The involution preserves the q -deformed Fock space $F_q[s]$ of multi charge s .

The following proposition shows that the action of B_m commutes with the bar involution.

Proposition 2.9 ([Ugl00]). *For $|\lambda; s\rangle \in F_q[s]$ and $m \in \mathbb{Z}_{>0}$, we have*

$$\overline{B_{-m}|\lambda; s\rangle} = B_{-m}\overline{|\lambda; s\rangle}. \quad (11)$$

2.7 The dominance order

We define a partial ordering $|\lambda; s\rangle \geq |\mu; s\rangle$.

Definition 2.10. *Let $|\lambda; s\rangle = u_{k_1} \wedge u_{k_2} \wedge \cdots$ and $|\mu; s\rangle = u_{g_1} \wedge u_{g_2} \wedge \cdots$. We define $|\lambda; s\rangle \geq |\mu; s\rangle$ if $|\lambda| = |\mu|$ and*

$$\sum_{j=1}^r k_j \geq \sum_{j=1}^r g_j \quad (\text{for all } r = 1, 2, 3, \dots). \quad (12)$$

Example 2.11. *Let $n = \ell = 2$, $s = (1, -1)$, $\lambda = ((1, 1), \emptyset)$, and $\mu = (\emptyset, (2))$. Then, $|\lambda; s\rangle = u_2 \wedge u_1 \wedge u_{-1} \wedge u_{-3} \wedge \cdots$ and $|\mu; s\rangle = u_3 \wedge u_1 \wedge u_{-2} \wedge u_{-3} \wedge \cdots$. Thus, $|\mu; s\rangle$ is greater than $|\lambda; s\rangle$.*

We define a matrix $(a_{\lambda, \mu}(q))_{\lambda, \mu}$ by

$$\overline{|\lambda; s\rangle} = \sum_{\mu} a_{\lambda, \mu}(q) |\mu; s\rangle. \quad (13)$$

Then the matrix $(a_{\lambda, \mu}(q))_{\lambda, \mu}$ is unitriangular with respect to \geq , that is

$$\begin{cases} \text{(a)} & \text{if } a_{\lambda, \mu}(q) \neq 0, \text{ then } |\lambda; s\rangle \geq |\mu; s\rangle, \\ \text{(b)} & a_{\lambda, \lambda}(q) = 1. \end{cases} \quad (14)$$

(see the identity (29) for the detail.)

Thus, by the standard argument, the unitriangularity implies the following theorem.

Theorem 2.12. [Ugl00, Theorem 3.25] *There exist unique bases $\{G^+(\lambda; s) \mid \lambda \in \Pi^\ell\}$ and $\{G^-(\lambda; s) \mid \lambda \in \Pi^\ell\}$ of $F_q[s]$ such that*

$$\begin{aligned} \text{(i)} \quad & \overline{G^+(\lambda; s)} = G^+(\lambda; s) \quad , \quad \overline{G^-(\lambda; s)} = G^-(\lambda; s) \\ \text{(ii)} \quad & G^+(\lambda; s) \equiv |\lambda; s\rangle \pmod{q \mathcal{L}^+} \quad , \quad G^-(\lambda; s) \equiv |\lambda; s\rangle \pmod{q^{-1} \mathcal{L}^-} \\ \text{where} \quad & \mathcal{L}^+ = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{Q}[q] |\lambda; s\rangle \quad , \quad \mathcal{L}^- = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{Q}[q^{-1}] |\lambda; s\rangle. \end{aligned}$$

Definition 2.13. Define matrices $\Delta^+(q) = (\Delta_{\lambda,\mu}^+(q))_{\lambda,\mu}$ and $\Delta^-(q) = (\Delta_{\lambda,\mu}^-(q))_{\lambda,\mu}$ by

$$G^+(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^+(q) |\mu; s\rangle \quad , \quad G^-(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^-(q) |\mu; s\rangle. \quad (15)$$

The entries $\Delta_{\lambda,\mu}^{\pm}(q)$ are called *q-decomposition numbers*. Note that *q*-decomposition numbers $\Delta^{\pm}(q)$ depend on n, ℓ and s . The matrices $\Delta^+(q)$ and $\Delta^-(q)$ are also unitriangular with respect to \geq .

It is known [Ugl00, Theorem 3.26] that the entries of $\Delta^-(q)$ are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type A , and that they are polynomials in q with non-negative integer coefficients.

3 A *q*-analogue of the tensor product theorem of level one

In this section we review the *q*-analogue of tensor product theorem in the case of $\ell = 1$ [LT00].

3.1 V_{λ} and S_{λ}

Let p_m, h_m and s_{λ} be the power sum symmetric function of degree m , the complete symmetric function of degree m and Schur function, respectively. There are some well-known relationship among them.

$$h_m = \sum_{|\lambda|=m} \frac{1}{z_{\lambda}} p_{\lambda} \quad , \quad s_{\lambda} = \sum_{\mu} K_{\mu,\lambda}^{(-1)} h_{\mu} \quad , \quad (16)$$

where $K_{\mu,\lambda}^{(-1)}$ is the inverse Kostka number and for a partition $\mu = (1^{\alpha_1}, 2^{\alpha_2}, \dots)$, we define $z_{\mu} = \prod_{i \geq 1} i^{\alpha_i} \alpha_i!$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we define $B_{-\lambda}$ by

$$B_{-\lambda} = B_{-\lambda_1} B_{-\lambda_2} \cdots . \quad (17)$$

Definition 3.1. For $m \in \mathbb{Z}_{>0}$ and $\lambda \in \Pi$, we define operators V_m and S_{λ} on $F_q[s]$ as

$$V_m = \sum_{|\lambda|=m} \frac{1}{z_{\lambda}} B_{-\lambda} \quad , \quad S_{\lambda} = \sum_{\mu} K_{\mu,\lambda}^{(-1)} V_{\mu} \quad , \quad (18)$$

where $V_{\mu} = V_{\mu_1} V_{\mu_2} \cdots$ for $\mu = (\mu_1, \mu_2, \dots) \in \Pi$.

That is, we regard B_{-m} (resp. V_m, S_{λ}) as the power sum (resp. the complete symmetric function, Schur function). By Proposition 2.9, V_m and S_{λ} also commute the bar involution.

The action of V_m is combinatorially described as follows. An *n-ribbon* is a connected strip of n -cells which does not contain a 2×2 square; more precisely, an *n-ribbon* is a sequence of n cells $R = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$ such that (a_{i+1}, b_{i+1}) is either $(a_i + 1, b_i)$ or $(a_i, b_i - 1)$, for $i = 1, 2, \dots, n$. The *head* of R is the cell $\text{head}(R) = (a_1, b_1)$ and $\text{spin}_n(R) = \#\{1 \leq i < n \mid a_{i+1} = a_i + 1\}$ is the *n-spin* of R .

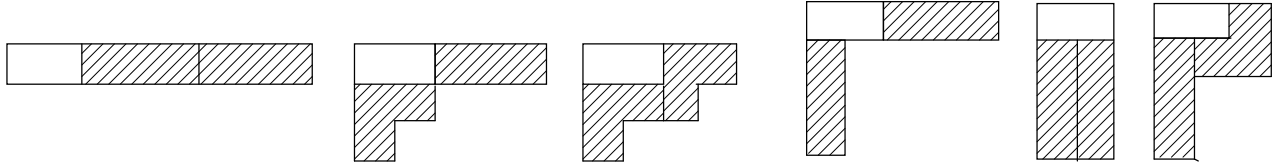
For partitions λ and μ , we write $\lambda \overset{m:n}{\rightsquigarrow} \mu$ if $\lambda \subset \mu$ and the skew diagram $\mu \setminus \lambda$ is a disjoint union of m n -ribbons such that the head of each ribbon is either in the first row of μ or is of the form (i, j) where $(i-1, j) \in \lambda$. Lascoux, Leclerc and Thibon call μ/λ an n -ribbon tableau of weight (m) and they note that there is a unique way of writing $\mu \setminus \lambda$ as a disjoint union of ribbons. Finally, if $\lambda \overset{m:n}{\rightsquigarrow} \mu$ then $\text{spin}_n(\mu/\lambda)$, the n -spin of μ/λ , is the sum of the n -spins of the ribbons in $\mu \setminus \lambda$.

Theorem 3.2 ([LT00] Theorem.6.7). *Let $m \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \Pi$. If $\ell = 1$, then*

$$V_m |\lambda\rangle = \sum_{\substack{\mu: n \\ \lambda \overset{m:n}{\rightsquigarrow} \mu}} (-q^{-1})^{\text{spin}_n(\mu/\lambda)} |\mu\rangle.$$

Example 3.3. *Let $n = 3$, $\ell = 1$, $m = 2$ and $\lambda = (2)$. Then,*

$$V_2 |(2); s\rangle = |(8); s\rangle - q^{-1} |(5, 2, 1); s\rangle + q^{-2} |(4, 3, 2); s\rangle + q^{-2} |(5, 1^3); s\rangle + q^{-4} |(2^4); s\rangle - q^{-3} |(3^2, 1^2); s\rangle.$$



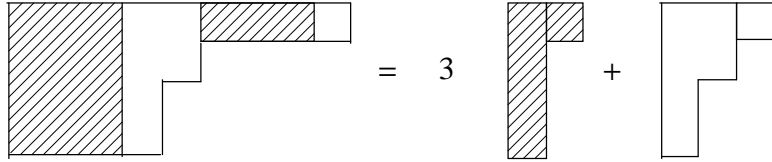
3.2 A q -analogue of the tensor product theorem in the case of $\ell = 1$

Definition 3.4. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is n -restricted is

$$0 \leq \lambda_i - \lambda_{i+1} < n \quad \text{for all } i = 1, 2, \dots. \quad (19)$$

Definition 3.5. For $\lambda \in \Pi$, we define $\tilde{\lambda}, \check{\lambda} \in \Pi$ as $\tilde{\lambda}$ is n -restricted and $\lambda = \tilde{\lambda} + n\check{\lambda}$.

Example 3.6. Let $n = 3$ and $\lambda = (9, 5, 4, 4)$. Then $(9, 5, 4, 4) = 3 \cdot (2, 1, 1, 1) + (3, 2, 1, 1)$. Thus, $\tilde{\lambda} = (3, 2, 1, 1)$, $\check{\lambda} = (2, 1, 1, 1)$.



Theorem 3.7 (Leclerc-Thibon[LT00], Theorem 6.9). *Let $\lambda = \tilde{\lambda} + n\check{\lambda}$. Then*

$$G^-(\lambda) = S_{\check{\lambda}} G^-(\tilde{\lambda}). \quad (20)$$

Remark. Theorem 3.7 is a formal analogue of Lusztig's tensor product theorem. (see [LT00, §3.2].)

Example 3.8. In this example, we write $|\lambda; s\rangle$ simply λ .

(i) Let $n = 2$, $\lambda = (4) = \emptyset + n(2)$. Then,

$$G^-((4)) = S_{(2)} G^-(\emptyset) = V_{(2)} \emptyset = (4) - q^{-1}(3, 1) + q^{-2}(2, 2).$$

(ii) Let $n = 2$ and $\lambda = (2, 2) = \emptyset + n(1, 1)$. Then,

$$G^-((2, 2)) = S_{(1,1)} G^-(\emptyset) = (V_{(1,1)} - V_2) \emptyset = (2, 2) - q^{-1}(2, 1, 1) + q^{-2}(1^4).$$

4 A q -analogue of the tensor product theorem of higher levels

4.1 M -dominancy

Definition 4.1. For $M \in \mathbb{Z}_{\geq 0}$, $|\lambda; s\rangle$ is M -dominant if

$$s_i - s_{i+1} \geq M + |\lambda| \quad (21)$$

for all $i = 1, 2, \dots, \ell - 1$.

Definition 4.2. For $1 \leq i \leq l$, we define linear operators $B'_m[i]$ by

$$B'_{-m}[i] |(\lambda^{(1)}, \dots, \lambda^{(i)}, \dots, \lambda^{(l)}); s\rangle = |(\lambda^{(1)}, \dots, B_{-m} \lambda^{(i)}, \dots, \lambda^{(l)}); s\rangle, \quad (22)$$

where the right hand side is understood as

$$|(\lambda^{(1)}, \dots, B_{-m} \lambda^{(i)}, \dots, \lambda^{(l)}); s\rangle = \sum_{\mu} c_{\mu} |(\lambda^{(1)}, \dots, \mu, \dots, \lambda^{(l)}); s\rangle$$

if $B_{-m} |\lambda^{(i)}\rangle = \sum_{\mu} c_{\mu} |\mu\rangle$ in the q -deformed Fock space of level one.

Proposition 4.3 ([Ugl00], Proposition 5.3(i)). Let $m \in \mathbb{Z}_{>0}$. Suppose that $|\lambda; s\rangle$ is nm -dominant. Then,

$$B_{-m} |\lambda; s\rangle = \sum_{i=1}^l q^{(i-1)m} B'_{-m}[i] |\lambda; s\rangle. \quad (23)$$

4.2 A q -analogue of the tensor product theorem of higher levels

Definition 4.4. For $1 \leq j \leq \ell$, we define $B_{-m}[j]$ as follows.

$$B_{-m}[\ell] = B'_{-m}[\ell] \quad , \quad B_{-m}[j] = B'_{-m}[j] - q^{-m} B'_{-m}[j+1] \quad , \quad (1 \leq j \leq \ell-1). \quad (24)$$

For $1 \leq j \leq \ell$,

$$B_{-m}[j, \ell] = \sum_{i=j}^{\ell} q^{(i-j)m} B'_{-m}[i]. \quad (25)$$

Lemma 4.5. If $u = |\lambda; s\rangle$ is nm -dominant, then

$$B_{-m} u = \sum_{j=1}^{\ell} \frac{q^{jm} - q^{-jm}}{q^m - q^{-m}} B_{-m}[j] u. \quad (26)$$

Proof.

$$\begin{aligned} \sum_{j=1}^{\ell} \frac{q^{jm} - q^{-jm}}{q^m - q^{-m}} B_{-m}[j] &= \sum_{j=1}^{\ell} \frac{q^{jm} - q^{-jm}}{q^m - q^{-m}} B'_{-m}[j] - q^{-m} \sum_{j=1}^{\ell-1} \frac{q^{jm} - q^{-jm}}{q^m - q^{-m}} B'_{-m}[j+1] \\ &= B'_{-m}[1] + \sum_{j=2}^{\ell} \left(\frac{q^{jm} - q^{-jm}}{q^m - q^{-m}} - q^{-m} \frac{q^{(j-1)m} - q^{-(j-1)m}}{q^m - q^{-m}} \right) B'_{-m}[j] \\ &= B'_{-m}[1] + \sum_{j=2}^{\ell} q^{(j-1)m} B'_{-m}[j]. \end{aligned}$$

Hence, the assertion follows from Proposition 4.3. □

The next proposition is the key proposition in this paper and will be proved in section 5.

Proposition 4.6. Let $m \in \mathbb{Z}_{>0}$ and $1 \leq i \leq \ell$. If $u = |\lambda; s\rangle$ is nm -dominant, then

- (i) $\overline{B_{-m}[j]} u = B_{-m}[j] \bar{u}$.
- (ii) $\overline{B_{-m}[j, \ell]} u = B_{-m}[j, \ell] \bar{u}$.

We define $V'_k[i]$, $V_k[i]$, $S'_\lambda[i]$, $S_\lambda[i]$ in the same fashion in Definition 3.1.

Definition 4.7. For $1 \leq i \leq \ell$, we define $V'_k[i]$, $V_k[i]$, $S'_\lambda[i]$, $S_\lambda[i]$ as follows.

$$\begin{aligned} V'_m[i] &= \sum_{|\lambda|=m} \frac{1}{z_\lambda} B'_{-\lambda}[i] \quad , \quad V_m[i] = \sum_{|\lambda|=m} \frac{1}{z_\lambda} B_{-\lambda}[i] \quad , \\ S'_\lambda[i] &= \sum_{\mu} K_{\mu, \lambda}^{(-1)} V'_\mu[i] \quad , \quad S_\lambda[i] = \sum_{\mu} K_{\mu, \lambda}^{(-1)} V_\mu[i] \quad . \end{aligned}$$

Remarks. (i). From the definition of $B'_{-m}[i]$, the operator $V'_m[i]$ (resp. $S'_\mu[i]$) acts on the i -th component of $|\lambda; s\rangle$ in the same way as V_m (resp. S_μ) in the case of $\ell = 1$. If $n = 2$, for example,

$$V'_m[1] |(\lambda^{(1)}, \lambda^{(2)}); s\rangle = |(V_m \lambda^{(1)}, \lambda^{(2)}); s\rangle \quad , \quad S'_\mu[2] |(\lambda^{(1)}, \lambda^{(2)}); s\rangle = |(\lambda^{(1)}, S_\mu \lambda^{(2)}); s\rangle.$$

In particular, $V'_m[i]$ has a combinatorial expression gives in Theorem 3.2.

(ii). Since $B_{-m}[\ell] = B'_{-m}[\ell]$, we have $V_m[\ell] = V'_m[\ell]$ and $S_\lambda[\ell] = S'_\lambda[\ell]$.

The next lemma gives the formula that expresses $S_\lambda[i]$ in terms of $S'_\mu[i]$ and $S'_\nu[i+1]$. Therefore we can compute $S_\lambda[i]u$ from the calculation in the case of $\ell = 1$ if u is $n|\lambda|$ -dominant.

Lemma 4.8. For $1 \leq j < \ell$,

$$S_\lambda[j] = \sum_{\mu, \nu} (-q^{-1})^{|\nu|} \text{LR}_{\mu\nu}^\lambda S'_\mu[j] S'_{\nu'}[j+1] \quad , \quad (27)$$

where $\text{LR}_{\mu\nu}^\lambda$ is the Littlewood-Richardson coefficient and ν' means the transpose of the Young diagram ν .

Proof. Let $X = (X_1, X_2, \dots)$, $Y = (Y_1, Y_2, \dots)$ be two families of variables. Let $\Lambda(X|Y) = \Lambda(X) \otimes \Lambda(Y)$ be the ring consisting of all symmetric function in both X and Y . Let $\mathcal{B} = \mathbb{Q}[B'_{-m}[i] \mid m \in \mathbb{Z}_{\geq 0}, i = j, j+1]$ be the ring consisting of all the \mathbb{Q} -linear combinations of products of elements $B'_{-m}[j]$ and $B'_{-m}[j+1]$. We define the ring homomorphism $\iota: \Lambda(X|Y) \rightarrow \mathcal{B}$ by $\iota(p_m(X)) = B'_{-m}[j]$ and $\iota(p_m(Y)) = B'_{-m}[j+1]$.

We define an automorphism ω_{-Y} on $\Lambda(X|Y)$ as follows.

$$\omega_{-Y}(p_m(X)) = p_m(X) \quad , \quad \omega_{-Y}(p_m(Y)) = -p_m(Y) \quad (m = 1, 2, \dots).$$

If we denote $q^{-1}Y$ by $(q^{-1}Y_1, q^{-1}Y_2, \dots)$, then

$$\omega_{-Y}(p_m(X, q^{-1}Y)) = \omega_{-Y}(p_m(X) + p_m(q^{-1}Y)) = \omega_{-Y}(p_m(X)) + q^{-m} \omega_{-Y}(p_m(Y)) = p_m(X) - q^{-m} p_m(Y)$$

Hence we have $B_{-m}[j] = \iota(\omega_{-Y}(p_m(X, q^{-1}Y)))$, and it follows from Definition 3.1 that

$$S_\lambda[j] = \iota(\omega_{-Y}(s_\lambda(X, q^{-1}Y))), \quad (28)$$

where $s_\lambda(X, q^{-1}Y)$ is the supersymmetric Schur function with variable $(X, q^{-1}Y) = (X_1, X_2, \dots, q^{-1}Y_1, q^{-1}Y_2, \dots)$.

Now we compute $\omega_{-Y}(s_\lambda(X, q^{-1}Y))$. Note that

$$\begin{aligned} \omega_{-Y}(h_k(Y)) &= \omega_{-Y}\left(\sum_{|\lambda|=k} \frac{1}{z_\lambda} p_\lambda(Y)\right) \\ &= \sum_{|\lambda|=k} \frac{(-1)^{l(\lambda)}}{z_\lambda} p_\lambda(Y) \\ &= (-1)^k \sum_{|\lambda|=k} \frac{(-1)^{k-l(\lambda)}}{z_\lambda} p_\lambda(Y) \\ &= (-1)^k e_k(Y), \end{aligned}$$

where e_k is the elementary symmetric function of degree k . Hence,

$$\omega_{-Y}(s_\nu(Y)) = (-1)^{|\nu|} s_{\iota_\nu}(Y).$$

From the well-known formula

$$s_\lambda(X, Y) = \sum_{\mu, \nu} \text{LR}_{\mu\nu}^\lambda s_\mu(X) s_\nu(Y),$$

we obtain

$$\begin{aligned} \omega_{-Y}(s_\lambda(X, q^{-1}Y)) &= \sum_{\mu, \nu} (-1)^{|\nu|} \text{LR}_{\mu\nu}^\lambda s_\mu(X) s_{\iota_\nu}(q^{-1}Y) \\ &= \sum_{\mu, \nu} (-q^{-1})^{|\nu|} \text{LR}_{\mu\nu}^\lambda s_\mu(X) s_{\iota_\nu}(Y). \end{aligned}$$

Therefore (27) follows from (28), $S'_\lambda[j] = \iota(s_\mu(X))$ and $S'_{\iota_\nu}[j+1] = \iota(s_{\iota_\nu}(Y))$. \square

Theorem 4.9. *Let $1 \leq j \leq \ell$ and $\lambda \in \Pi$. If $|\mu; s\rangle$ is $n|\lambda|$ -dominant and $\mu^{(j)}$ is n -restricted,*

$$S_\lambda[j] G^-(\mu; s) = G^-(\mu^{(1)}, \dots, \mu^{(j)} + n\lambda, \dots, \mu^{(\ell)}; s).$$

Proof. By definition of the basis G^- , we have to prove that $F = S_\lambda[j] G^-(\mu; s)$ satisfies

$$\overline{F} = F \quad \text{and} \quad F \equiv |(\mu^{(1)}, \dots, \mu^{(i)} + n\lambda, \dots, \mu^{(\ell)}); s\rangle \pmod{q^{-1}\mathcal{L}^-}.$$

The first property is clear by Proposition 4.6. Indeed, $S_\lambda[j]$ is a \mathbb{Q} -linear combination of products of elements $B_{-m}[j]$. To prove second property, we observe that by Theorem 3.2 for all $\rho \in \Pi^\ell$ and $m \in \mathbb{N}$, $V'_m[j] |\rho; s\rangle \in \mathcal{L}^-$. Thus, $S'_\lambda[j] |\rho; s\rangle \in \mathcal{L}^-$ since $S'_\lambda[j]$ is a \mathbb{Q} -linear combination of products of elements $V'_m[j]$.

Note also the following lemma.

Lemma 4.10 ([LT00] Proof of Theorem 6.9). *Let $\ell = 1$. Let λ, ρ be two partitions such that ρ is n -restricted. Then,*

$$S_\lambda |\rho\rangle \equiv |\rho + n\lambda\rangle \pmod{q^{-1}\mathcal{L}^-}.$$

From the lemma, we obtain

$$\begin{aligned} F = S_\lambda[j] G^-(\mu; s) &\equiv \left(\sum_{\rho, \kappa} (-q^{-1})^{|\kappa|} \text{LR}_{\rho\kappa}^\lambda S'_\rho[j] S'_{\iota_\kappa}[j+1] \right) G^-(\mu; s) \quad (\text{By lemma 4.8}) \\ &\equiv S'_\lambda[j] G^-(\mu; s) \quad (\text{By } S'_{\iota_\kappa}[j+1] G^-(\mu; s) \in \mathcal{L}^-) \\ &\equiv S'_\lambda[j] |\mu; s\rangle \quad (\text{By } G^-(\mu; s) \equiv |\mu; s\rangle) \\ &\equiv |(\mu^{(1)}, \dots, \mu^{(i)} + n\lambda, \dots, \mu^{(\ell)}); s\rangle \quad (\text{By lemma 4.10}). \end{aligned}$$

\square

Definition 4.11. For $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)}) \in \Pi^\ell$, we define $\widetilde{\lambda}, \check{\lambda} \in \Pi^\ell$ by

$$\widetilde{\lambda} = (\widetilde{\lambda}^{(1)}, \widetilde{\lambda}^{(2)}, \dots, \widetilde{\lambda}^{(\ell)}) \quad , \quad \check{\lambda} = (\check{\lambda}^{(1)}, \check{\lambda}^{(2)}, \dots, \check{\lambda}^{(\ell)}) \quad ,$$

where $\lambda^{(i)} = \widetilde{\lambda}^{(i)} + n\check{\lambda}^{(i)}$ and $\widetilde{\lambda}^{(i)}$ is n -restricted. (See Definition 3.5.) And we define

$$S_\lambda = \prod_{i=1}^{\ell} S_{\lambda^{(i)}}[i] = S_{\lambda^{(1)}}[1] S_{\lambda^{(2)}}[2] \cdots S_{\lambda^{(\ell)}}[\ell].$$

Now we can state our main result, which is a higher level version of the q -analogue of Theorem 3.7.

Theorem 4.12. If $|\lambda; s\rangle$ is 0-dominant, then

$$G^-(\lambda; s) = S_{\check{\lambda}} G^-(\widetilde{\lambda}; s).$$

Proof. Note that for any partition μ , $\mu = n\check{\mu} + \widetilde{\mu}$. Since $|\lambda; s\rangle$ is 0-dominant, for any $1 \leq i < \ell$ and $1 \leq j \leq \ell$,

$$s_i - s_{i+1} \geq |\lambda| = \sum_{k=1}^{\ell} |\lambda^{(k)}| \geq \sum_{k=1}^{j-1} |\widetilde{\lambda}^{(k)}| + |\lambda^{(j)}| + \sum_{k=j+1}^{\ell} |\lambda^{(k)}| = n|\check{\lambda}^{(j)}| + \sum_{k=1}^j |\widetilde{\lambda}^{(k)}| + \sum_{k=j+1}^{\ell} |\lambda^{(k)}|.$$

Hence $|(\widetilde{\lambda}^{(1)}, \dots, \widetilde{\lambda}^{(j)}, \lambda^{(j+1)}, \dots, \lambda^{(\ell)}; s)\rangle$ is $n|\check{\lambda}^{(j)}|$ -dominant for any $1 \leq j \leq \ell$. Thus, by applying Theorem 4.9 repeatedly,

$$\begin{aligned} S_{\check{\lambda}} G^-(\widetilde{\lambda}; s) &= S_{\check{\lambda}^{(1)}}[1] \cdots S_{\check{\lambda}^{(\ell-1)}}[\ell-1] S_{\check{\lambda}^{(\ell)}}[\ell] G^-(\widetilde{\lambda}^{(1)}, \dots, \widetilde{\lambda}^{(\ell-1)}, \widetilde{\lambda}^{(\ell)}; s) \\ &= S_{\check{\lambda}^{(1)}}[1] \cdots S_{\check{\lambda}^{(\ell-1)}}[\ell-1] G^-(\widetilde{\lambda}^{(1)}, \dots, \widetilde{\lambda}^{(\ell-1)}, \lambda^{(\ell)}; s) \\ &= \cdots = G^-(\lambda^{(1)}, \dots, \lambda^{(\ell-1)}, \lambda^{(\ell)}; s) = G^-(\lambda; s). \end{aligned}$$

□

Example 4.13. In this example, we write $|\lambda; s\rangle$ simply λ .

(i) Let $n = l = 2$, $\lambda = ((2, 2), \emptyset)$ and $s = (2, -2)$. Then, $|\lambda; s\rangle$ is 0-dominant and $(2, 2) = \emptyset + n(1, 1)$.

$$\begin{aligned} G^-((2, 2), \emptyset) &= S_{((1, 1), \emptyset)}(\emptyset, \emptyset) \\ &= S_{(1, 1)}[1](\emptyset, \emptyset) \\ &= (S'_{(1, 1)}[1] - q^{-1}S'_{(1)}[1]S'_{(1)}[2] + q^{-2}S'_{(2)}[2])(\emptyset, \emptyset) \quad (\text{Lemma 4.8}) \\ &= (S_{(1, 1)}\emptyset, \emptyset) - q^{-1}(S_{(1)}\emptyset, S_{(1)}\emptyset) + q^{-2}(\emptyset, S_{(2)}\emptyset) \\ &= ((2, 2) - q^{-1}(2, 1, 1) + q^{-2}(1^4), \emptyset) - q^{-1}((2) - q^{-1}(1, 1), (2) - q^{-1}(1, 1)) \\ &\quad + q^{-2}(\emptyset, (4) - q^{-1}(3, 1) + q^{-2}(2, 2)) \\ &= ((2, 2), \emptyset) - q^{-1}((2, 1, 1), \emptyset) + q^{-2}((1^4), \emptyset) - q^{-1}((2), (2)) + q^{-2}((2), (1, 1)) + q^{-2}((1, 1), (2)) \\ &\quad - q^{-3}((1, 1), (1, 1)) + q^{-2}(\emptyset, (4)) - q^{-3}(\emptyset, (3, 1)) + q^{-4}(\emptyset, (2, 2)). \end{aligned}$$

(ii) Let $n = l = 2$ and $s = (3, -3)$. Then, $((2), (2, 2))$ is 0-dominant, $(2) = \emptyset + n(1)$ and $(2, 2) = \emptyset + n(1, 1)$.

$$\begin{aligned}
& G^-((2), (2, 2)) \\
&= S_{((1), (1, 1))}(\emptyset, \emptyset) \\
&= S_{(1)}[1] S_{(1, 1)}[2](\emptyset, \emptyset) \\
&= (S'_{(1)}[1] - q^{-1} S'_{(1)}[2]) S'_{(1, 1)}[2](\emptyset, \emptyset) \quad (\text{Lemma.4.8}) \\
&= (S'_{(1)}[1] - q^{-1} S'_{(1)}[2])(\emptyset, S_{(1, 1)}\emptyset) \\
&= (S_{(1)}\emptyset, S_{(1, 1)}\emptyset) - q^{-1}(\emptyset, S_{(1)}S_{(1, 1)}\emptyset) \\
&= (S_{(1)}\emptyset, S_{(1, 1)}\emptyset) - q^{-1}(\emptyset, S_{(2, 1)}\emptyset) - q^{-1}(\emptyset, S_{(1^3)}\emptyset) \\
&= ((2) - q^{-1}, (2, 2) - q^{-1}(2, 1, 1) + q^{-2}(1^4)) \\
&\quad - q^{-1}(\emptyset, (4, 2) - q^{-1}(4, 1, 1) - q^{-1}(3, 3) + q^{-2}(3, 1^3) - q^3(2^3) + q^{-4}(2^2, 1^2)) \\
&\quad - q^{-1}(\emptyset, (2^3) - q^{-1}(2^2, 1^2) + q^{-2}(2, 1^4) - q^{-3}(1^6)) \\
&= ((2), (2, 2)) - q^{-1}((2), (2, 1, 1)) + q^{-2}((2), (1^4)) - q^{-1}((1, 1), (2, 2)) + q^{-2}((1, 1), (2, 1, 1)) - q^{-3}((1, 1), (1^4)) \\
&\quad - q^{-1}(\emptyset, (4, 2)) + q^{-2}(\emptyset, (4, 1, 1)) + q^{-2}(\emptyset, (3, 3)) - q^{-3}(\emptyset, (3, 1^3)) - (q^{-1} + q^{-3})(\emptyset, (2^3)) \\
&\quad + (q^{-2} + q^{-4})(\emptyset, (2^2, 1^2)) - q^{-3}(\emptyset, (2, 1^4)) + q^{-4}(\emptyset, (1^6)).
\end{aligned}$$

5 Proof of Proposition 4.6

We prove Proposition 4.6 by using (infinite) q -wedge product. Fix a sufficiently large integer r so that for every ordered q -wedge product appearing in our argument, all of the components after r -th factor are consecutive. We are able to truncate q -wedge products at the first r parts. See [Ugl00, §4] for detail.

5.1 q -wedges and straightening rules

In this section, we review the straightening rules [Ugl00] to prove our main results.

Proposition 5.1 ([Iij12] Proposition 4.2 ,see also [Ugl00] Proposition 3.16). *For unequal integers k_1, k_2 , let c_j, d_j, m_j be the unique integers satisfying $k_j = c_j + n(d_j - 1) - n\ell m_j$, $1 \leq c_j \leq n$ and $1 \leq d_j \leq \ell$, ($j = 1, 2$). Then,*

$$\begin{aligned}
u_{c_2 - nm_2}^{(d_2)} \wedge u_{c_1 - nm_1}^{(d_1)} &= (-q^{-1})^{\delta_{d_1=d_2}} q^\alpha u_{c_1 - nm_1}^{(d_1)} \wedge u_{c_2 - cm_2}^{(d_2)} \\
&\quad + \text{sgn}(m) (-q^{-1})^{\delta_{d_1=d_2}} (q - q^{-1}) \sum_{j=\beta}^{|m_1 - m_2| - \gamma} u_{c_2 - nm_1 - \text{sgn}(m)nj}^{(d_1)} \wedge u_{c_1 - nm_2 + \text{sgn}(m)nj}^{(d_2)}. \quad (29)
\end{aligned}$$

where

$$\text{sgn}(m) = \begin{cases} 1 & \text{if } m_1 < m_2 \\ -1 & \text{if } m_1 > m_2 \\ 0 & \text{if } m_1 = m_2 \end{cases}, \quad \alpha = \begin{cases} 1 & \text{if } c_1 = c_2 \text{ and } k_1 > k_2 \\ -1 & \text{if } c_1 = c_2 \text{ and } k_1 < k_2 \\ 0 & \text{if } c_1 \neq c_2 \end{cases},$$

$$\delta_{d_1=d_2} = \begin{cases} 1 & \text{if } d_1 = d_2 \\ 0 & \text{if } d_1 \neq d_2 \end{cases}, \quad \beta = \begin{cases} 0 & \text{if } c_1 > c_2, m_1 < m_2 \text{ or } c_1 < c_2, m_1 > m_2 \\ 1 & \text{if otherwise} \end{cases},$$

and

$$\gamma = \begin{cases} 1 & \text{if } d_1 < d_2, m_1 < m_2 \text{ or } d_1 > d_2, m_1 > m_2 \\ 0 & \text{if } d_1 > d_2, m_1 < m_2 \text{ or } d_1 < d_2, m_1 > m_2 \end{cases}.$$

Remarks. Note that the identity (29) depends only on the inequality relationship between d_1 and d_2 (c_1 and c_2). It is independent of ℓ .

Corollary 5.2. Suppose $u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_2)}$ is expressed by straightening rule as

$$u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_2)} = \sum_{g_1, g_2} \alpha(g_1, g_2) u_{g_2}^{(d_2)} \wedge u_{g_1}^{(d_1)},$$

where $\alpha(g_1, g_2) \in \mathbb{Q}(q)$. Then,

(i) $k_1 + k_2 = g_1 + g_2$.

(ii) If $k_1 < k_2$, then $k_1 \leq g_1 \leq k_2$ and $k_1 \leq g_2 \leq k_2$. If $k_1 > k_2$, then $k_1 \geq g_1 \geq k_2$ and $k_1 \geq g_2 \geq k_2$.

(iii) Let $k_i = c_i - nm_i$ and $g_i = c'_i - nm'_i$ ($i = 1, 2$) where c_i, c'_i and m_i, m'_i are defined in Definition 2.4. Then, $\{c_1, c_2\} = \{c'_1, c'_2\}$.

Corollary 5.3. Let k, m, g be three integers such that $m \geq 0$ and $k \geq g + nm$. If $u_k^{(d_1)} \wedge u_g^{(d_2)}$ and $u_k^{(d_1)} \wedge u_{g+nm}^{(d_2)}$ is expressed by straightening rule as

$$u_k^{(d_1)} \wedge u_g^{(d_2)} = \sum_{j=0}^{k-g} C_j(q) u_{g+j}^{(d_2)} \wedge u_{k-j}^{(d_1)}, \quad u_k^{(d_1)} \wedge u_{g+nm}^{(d_2)} = \sum_{j=0}^{k-g-nm} C'_j(q) u_{g+nm+j}^{(d_2)} \wedge u_{k-j}^{(d_1)},$$

where $C_j(q), C'_j(q) \in \mathbb{Q}(q)$, then $C_j(q) = C'_j(q)$ for all $0 \leq j \leq k - g - nm$.

Definition 5.4. Let

$$u = u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_2)} \wedge \cdots \wedge u_{k_r}^{(d_r)}, \quad k_a = c_a - nm_a, \quad (a = 1, 2, \dots, r) \quad \text{and}$$

$$v = u_{g_1}^{(d'_1)} \wedge u_{g_2}^{(d'_2)} \wedge \cdots \wedge u_{g_t}^{(d'_t)}, \quad g_b = c'_b - nm'_b, \quad (b = 1, 2, \dots, t).$$

and suppose that $d_a \neq d'_b$ for all $a \in \{1, \dots, r\}$ and $b \in \{1, \dots, t\}$. Then we define $\xi(u, v)$ as

$$\xi(u, v) = \#\{(a, b) \mid c_a = c'_b, \quad u_{k_a}^{(d_a)} < u_{g_b}^{(d'_b)}\}. \quad (30)$$

Lemma 5.5 ([Iij12] Lemma 4.8, (see [Ugl00] Lemma 5.19)). *Let $a \in \mathbb{Z}$, $t \in \mathbb{Z}_{\geq 0}$, $1 \leq i \leq \ell$, and $1 \leq j \leq \ell$.*

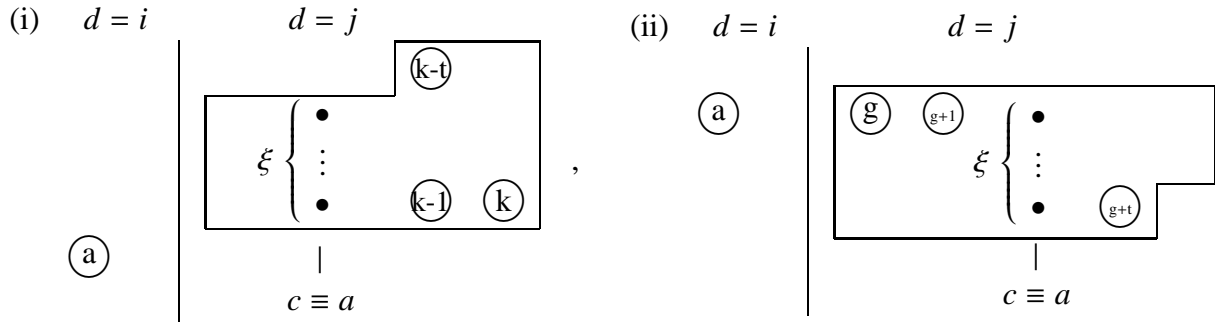
(i). *Let $u_k^{(j)}$ be the maximal element such that $u_k^{(j)} < u_a^{(i)}$. Let $u_{[k,k-t]}^{(j)} = u_k^{(j)} \wedge u_{k-1}^{(j)} \wedge \cdots \wedge u_{k-t}^{(j)}$. Then,*

$$u_a^{(i)} \wedge u_{[k,k-t]}^{(j)} = q^{-\xi(u_{[k,k-t]}^{(j)}, u_a^{(i)})} u_{[k,k-t]}^{(j)} \wedge u_a^{(i)}.$$

(ii). *Let $u_g^{(j)}$ be the minimal element such that $u_g^{(j)} > u_a^{(i)}$. Let $u_{[g+t,g]}^{(j)} = u_{g+t}^{(j)} \wedge u_{g+t-1}^{(j)} \wedge \cdots \wedge u_g^{(j)}$. Then,*

$$u_a^{(i)} \wedge u_{[g+t,g]}^{(j)} = q^{\xi(u_a^{(i)}, u_{[g+t,g]}^{(j)})} u_{[g+t,g]}^{(j)} \wedge u_a^{(i)}.$$

In the abacus presentation, $u_a^{(i)}$, $u_{[k,k-t]}^{(j)}$ and $u_{[g+t,g]}^{(j)}$ look as follows.



where the boxed region means that all positions are occupied by beads.

Throughout this section, $s = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$ is a fixed multi charge.

5.2 A deletion lemma for finite q -wedges

Definition 5.6. *Let $1 \leq d_1, d_2 \leq \ell$. Let $v^{(d_1)} = u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_1)} \wedge \cdots \wedge u_{k_r}^{(d_1)}$ and $w^{(d_2)} = u_{g_1}^{(d_2)} \wedge u_{g_2}^{(d_2)} \wedge \cdots \wedge u_{g_r}^{(d_2)}$ be two simple finite q -wedges.*

(i) *The size of $v^{(d_1)}$ for s_{d_1} is*

$$|v^{(d_1)}|_{s_{d_1}} = |u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_1)} \wedge \cdots \wedge u_{k_r}^{(d_1)}|_{s_{d_1}} = \sum_{i=1}^r |k_i - s_{d_1} + i|.$$

(ii) *For a non-negative integer M , the pair $(v^{(d_1)}, w^{(d_2)}, s_{d_1}, s_{d_2})$ is M -dominant if*

$$|v^{(d_1)}|_{s_{d_1}} + |w^{(d_2)}|_{s_{d_2}} + M \leq s_{d_1} - s_{d_2}.$$

Example 5.7. (i) Let $s_1 = 2$, $s_2 = -4$, $v^{(1)} = u_4^{(1)} \wedge u_2^{(1)} \wedge u_1^{(1)} \wedge u_{-1}^{(1)} \wedge u_{-2}^{(1)} \wedge u_{-3}^{(1)}$ and $w^{(2)} = u_{-2}^{(2)}$. Then $|v^{(1)}|_{s_1} = 4$ and $|w^{(2)}|_{s_2} = 2$. Thus, $|v^{(1)}|_{s_1} + |w^{(2)}|_{s_2} \leq s_1 - s_2$. Therefore $(v^{(1)}, w^{(2)}, s_1, s_2)$ is 0-dominant.

(ii) Let r be a positive integer and λ be a partition whose length is less than r . For any $1 \leq j \leq \ell$, put

$$v_\lambda^{(j)} = u_{\lambda_1+s_j}^{(j)} \wedge u_{\lambda_2+s_{j-1}}^{(j)} \wedge \cdots \wedge u_{\lambda_r+s_{j-r+1}}^{(j)}.$$

Then, $|v_\lambda^{(j)}|_{s_j} = |\lambda|$.

Remarks. (i). Let $v^{(j)} = u_{k_1}^{(j)} \wedge u_{k_2}^{(j)} \wedge \cdots \wedge u_{k_r}^{(j)}$ and $w^{(j)} = u_{g_1}^{(j)} \wedge u_{g_2}^{(j)} \wedge \cdots \wedge u_{g_r}^{(j)}$ be two simple finite q -wedges such that the length of $v^{(j)}$ is equal to that of $w^{(j)}$. Then, the size of $v^{(j)}$ for s_j is equal to that of $w^{(j)}$ if and only if $\sum_{i=1}^r k_i = \sum_{i=1}^r g_i$.

(ii) From Corollary 5.2 (i), straightening rule preserves $|v^{(d_1)}|_{s_{d_1}} + |w^{(d_2)}|_{s_{d_2}}$. That is, if $x^{(d_1)} \wedge y^{(d_2)}$ appears in a linear expansion of $w^{(d_2)} \wedge v^{(d_1)}$, then $|v^{(d_1)}|_{s_{d_1}} + |w^{(d_2)}|_{s_{d_2}} = |x^{(d_1)}|_{s_{d_1}} + |y^{(d_2)}|_{s_{d_2}}$.

In particular, for a non-negative integer M , if the pair $(v^{(d_1)}, w^{(d_2)}, s_{d_1}, s_{d_2})$ is M -dominant, then the pair $(x^{(d_1)}, y^{(d_2)}, s_{d_1}, s_{d_2})$ is also M -dominant.

Lemma 5.8. Let $1 \leq j \leq \ell$ and $v^{(j)} = u_{k_1}^{(j)} \wedge u_{k_2}^{(j)} \wedge \cdots \wedge u_{k_r}^{(j)}$ be a simple finite q -wedge. Suppose that $\min\{k_1, k_2, \dots, k_r\} \geq s_j - r + 1$ and $|v^{(j)}|_{s_j} < 0$, then $v^{(j)} = 0$.

Proof. If $v^{(j)} \neq 0$, then an ordered simple finite q -wedge $v' = u_{k'_1}^{(j)} \wedge u_{k'_2}^{(j)} \wedge \cdots \wedge u_{k'_r}^{(j)}$ appears in the linear expansion of $v^{(j)}$. Then from Corollary 5.2 (ii), we have $k'_1 > k'_2 > \cdots > k'_r \geq s_j - r + 1$. Thus $|v'|_{s_j} \geq 0$.

On the other hand, from Corollary 5.2 (i) (see the previous Remark (ii)), $|v^{(1)}|_{s_1} = |v'|_{s_1}$. This is a contradiction. \square

Lemma 5.9. Let $M, \gamma \in \mathbb{Z}_{\geq 0}$ such that $0 \leq \gamma \leq M$. Let $1 \leq d_1, d_2 \leq \ell$, $t \in \mathbb{Z}$ and $v^{(d_1)} = u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_1)} \wedge \cdots \wedge u_{k_r}^{(d_1)}$. If $(v^{(d_1)}, u_t^{(d_2)}, s_{d_1}, s_{d_2})$ is M -dominant, then

$$|u_{t+\gamma}^{(d_1)} \wedge v^{(d_1)}|_{s_{d_1}+1} < 0.$$

Moreover, if $\min\{t + \gamma, k_1, k_2, \dots, k_r\} \geq s_{d_1} - r$, then $u_{t+\gamma}^{(d_1)} \wedge v^{(d_1)} = 0$.

Proof. For simplicity, we assume that $d_1 = 1$ and $d_2 = 2$.

$$\begin{aligned} |u_{t+\gamma}^{(1)} \wedge v^{(1)}|_{s_1+1} &= t + \gamma - (s_1 + 1) + \sum_{i=1}^r (k_i - (s_1 + 1) + (i + 1) - 1) \\ &= t - s_1 + \gamma - 1 + |v^{(1)}|_{s_1} \\ &= |u_t^{(2)}|_{s_2} + |v^{(1)}|_{s_1} + s_2 - s_1 + \gamma - 1 \quad (\text{By } |u_t^{(2)}|_{s_2} = t - s_2) \\ &\leq s_1 - s_2 - M + s_2 - s_1 + \gamma - 1 \quad ((v^{(1)}, u_t^{(2)}, s_1, s_2) \text{ is } M\text{-dominant}) \\ &= \gamma - M - 1 < 0. \quad (\gamma \leq M) \end{aligned}$$

The second statement follows from the first statement and Lemma 5.8. \square

Lemma 5.10. Let $v^{(j)} = u_{k_1}^{(j)} \wedge u_{k_2}^{(j)} \wedge \cdots \wedge u_{k_r}^{(j)}$ be an ordered q -wedge such that $k_r \geq s_j - r + 1$. Then, for any $1 \leq a \leq r$,

$$|u_{k_a}^{(j)} \wedge u_{k_{a+1}}^{(j)} \wedge \cdots \wedge u_{k_r}^{(j)}|_{s_j-a+1} \leq |v^{(j)}|_{s_j}.$$

In particular, for a non-negative integer M , if $(v^{(d_1)}, w^{(d_2)}, s_{d_1}, s_{d_2})$ is M -dominant, then for all $1 \leq a \leq r$, $(u_{k_a}^{(d_1)} \wedge u_{k_{a+1}}^{(d_1)} \wedge \cdots \wedge u_{k_r}^{(d_1)}, w^{(d_2)}, s_{d_1} - i + 1, s_{d_2})$ is also M -dominant.

Proof. Since $v^{(j)}$ is ordered and $k_r \geq s_j - r + 1$, $k_1 > k_2 > \cdots > k_r \geq s_j - r + 1$. Hence, for any $1 \leq a \leq r$, $k_a \geq s_j - a + 1$. Thus,

$$\begin{aligned} |u_{k_a}^{(j)} \wedge u_{k_{a+1}}^{(j)} \wedge \cdots \wedge u_{k_r}^{(j)}|_{s_j-a+1} &= \sum_{i=a}^r (k_i - (s_j - a + 1) + (i - a + 1) - 1) \\ &= \sum_{i=a}^r (k_i - s_j + i - 1) \\ &\leq \sum_{i=1}^r (k_i - s_j + i - 1) = |v^{(j)}|_{s_j} \end{aligned}$$

The second statement is clear. □

5.3 A exchange rule for dominant finite q -wedges

Definition 5.11. Let $1 \leq j \leq \ell$, $\lambda \in \Pi$ and k, g be two integers such that $k \geq g$. We define

$$\begin{aligned} u_{[k,g]}^{(j)} &= u_k^{(j)} \wedge u_{k-1}^{(j)} \wedge u_{k-2}^{(j)} \wedge \cdots \wedge u_{g+1}^{(j)} \wedge u_g^{(j)}, \\ v_{\lambda,r}^{(j)} &= u_{\lambda_1+s_j}^{(j)} \wedge u_{\lambda_2+s_j-1}^{(j)} \wedge \cdots \wedge u_{\lambda_r+s_j-r+1}^{(j)}. \end{aligned}$$

Lemma 5.12. Let $m \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \Pi$. Let t, t_0 be two integers such that $t \geq t_0$. Suppose that $|\lambda| + |u_t^{(d_2)}|_{s_{d_2}} \leq s_{d_1} - s_{d_2} - nm$. Put $r = s_{d_1} - t_0$ and $v^{(d_1)} = v_{\lambda,r}^{(d_1)}$. If $v^{(d_1)} \wedge u_t^{(d_2)}$ is expressed as

$$v^{(d_1)} \wedge u_t^{(d_2)} = \sum_{t', y} \alpha(t', y) u_{t'}^{(d_2)} \wedge y, \quad ,$$

where $t' \in \mathbb{Z}$, $y = u_{\mathbf{k}}^{(d_1)}$ ($\mathbf{k} \in \mathbb{Z}^r$) and $\alpha(t', y) \in \mathbb{Q}(q)$. Then,

$$v^{(d_1)} \wedge u_{t+nm}^{(d_2)} = q^{2m} \sum_{t', y} \alpha(t', y) u_{t'+nm}^{(d_2)} \wedge y \quad .$$

Example 5.13. Let $n = \ell = 2, m = 1$ and $s_1 = 2, s_2 = -4$. Let $\lambda = (1, 1)$ and $t = t_0 = -4$. Then, $r = 6$ and $v^{(1)} = v_{(1,1),6}^{(1)} = u_2^{(1)} \wedge u_1^{(1)} \wedge u_{[-1,-4]}^{(1)} = u_2^{(1)} \wedge u_1^{(1)} \wedge u_{-1}^{(1)} \wedge u_{-2}^{(1)} \wedge u_{-3}^{(1)} \wedge u_{-4}^{(1)}$.

Since $|\lambda| + |u_{-4}^{(2)}|_{s_2} = 2 + 0 \leq s_1 - s_2 - nm$, these satisfy the condition in Lemma 5.12. The expansions of $v^{(1)} \wedge u_{-4}^{(2)}$ and $v^{(1)} \wedge u_{-2}^{(2)}$ are

$$\begin{aligned} v^{(1)} \wedge u_{-4}^{(2)} &= q^{-1} u_{-4}^{(2)} \wedge u_2^{(1)} \wedge u_1^{(1)} \wedge u_{[-1,-4]}^{(1)} - (q - q^{-1}) u_{-3}^{(2)} \wedge u_2^{(1)} \wedge u_0^{(1)} \wedge u_{[-1,-4]}^{(1)} \\ &\quad + (q^2 - 1) u_{-2}^{(2)} \wedge u_1^{(1)} \wedge u_0^{(1)} \wedge u_{[-1,-4]}^{(1)}, \\ v^{(1)} \wedge u_{-2}^{(2)} &= q u_{-2}^{(2)} \wedge u_2^{(1)} \wedge u_1^{(1)} \wedge u_{[-1,-4]}^{(1)} - (q^3 - q) u_{-1}^{(2)} \wedge u_2^{(1)} \wedge u_0^{(1)} \wedge u_{[-1,-4]}^{(1)} \\ &\quad + (q^4 - q^2) u_0^{(2)} \wedge u_1^{(1)} \wedge u_0^{(1)} \wedge u_{[-1,-4]}^{(1)}. \end{aligned}$$

On the other hand, let $w^{(1)} = u_2^{(1)} \wedge u_1^{(1)}$. Then, the expansions of $w^{(1)} \wedge u_{-4}^{(2)}$ and $w^{(1)} \wedge u_{-2}^{(2)}$ are

$$\begin{aligned} w^{(1)} \wedge u_{-4}^{(2)} &= q^{-1} u_{-4}^{(2)} \wedge u_2^{(1)} \wedge u_1^{(1)} - (q - q^{-1}) u_{-3}^{(2)} \wedge u_2^{(1)} \wedge u_0^{(1)} + (q^2 - 1) u_{-2}^{(2)} \wedge u_1^{(1)} \wedge u_0^{(1)} \\ &\quad - (q - q^{-1}) u_{-1}^{(2)} \wedge u_2^{(1)} \wedge u_{-2}^{(1)} + (q^2 + 1) u_0^{(2)} \wedge u_1^{(1)} \wedge u_{-2}^{(1)}, \\ w^{(1)} \wedge u_{-2}^{(2)} &= q^{-1} u_{-2}^{(2)} \wedge u_2^{(1)} \wedge u_1^{(1)} - (q - q^{-1}) u_{-1}^{(2)} \wedge u_2^{(1)} \wedge u_0^{(1)} + (q^2 - 1) u_0^{(2)} \wedge u_1^{(1)} \wedge u_0^{(1)}. \end{aligned}$$

In general, the expansion of $w^{(1)} \wedge u_t^{(2)}$ has more terms than that of $w^{(1)} \wedge u_{t+nm}^{(2)}$. Lemma 5.12 asserts that $u_{[-1,-4]}^{(1)}$ deletes the "excessive" terms and the coefficient q^{2m} comes from the difference between $u_{[-1,-4]}^{(1)} \wedge u_{-4}^{(2)}$ and $u_{[-1,-4]}^{(1)} \wedge u_{-2}^{(2)}$.

(proof of Lemma 5.12). For simplicity, we assume that $d_1 = 1$ and $d_2 = 2$.

From $r = s_1 - t_0$ and $|\lambda| + |u_t^{(2)}|_{s_2} \leq s_1 - s_2 - nm$, we have $|\lambda| \leq r - nm - t + t_0$. Thus, $\lambda_r = \lambda_{r-1} = \dots = \lambda_{r-nm-t+t_0+1} = 0$ and $|\lambda| = |v^{(1)}|_{s_1}$. In particular, the pair $(v^{(1)}, u_t^{(2)}, s_1, s_2)$ is nm -dominant.

We divide $v^{(1)}$ into two parts, say

$$\begin{aligned} w^{(1)} &= u_{\lambda_1+s_1}^{(1)} \wedge u_{\lambda_2+s_1-1}^{(1)} \wedge \dots \wedge u_{\lambda_{r-nm-t+t_0}+s_1-(r-nm-t+t_0-1)}^{(1)} = v_{\lambda, r-nm-t+t_0}^{(1)}, \\ x^{(1)} &= u_{s_1-r+nm+t-t_0}^{(1)} \wedge u_{s_1-r+nm+t-t_0-1}^{(1)} \wedge \dots \wedge u_{s_1-r+1}^{(1)} = u_{[t+nm, t_0+1]}^{(1)}. \end{aligned}$$

It is clear that $v^{(1)} = w^{(1)} \wedge x^{(1)}$.

Let $c = \xi(u_{[t, t_0+1]}^{(1)}, u_t^{(2)})$. By Lemma 5.5, $x^{(1)} \wedge u_t^{(2)} = q^{c-m} u_t^{(2)} \wedge x^{(1)}$ and $x^{(1)} \wedge u_{t+nm}^{(2)} = q^{c+m} u_t^{(2)} \wedge x^{(1)}$, since $\xi(u_{[t+nm, t+1]}^{(1)}, u_t^{(2)}) = \xi(u_{t+nm}^{(2)}, u_{[t+nm, t+1]}^{(1)}) = m$. Hence, the following claim deduces Lemma 5.12.

Claim 1. If $w^{(1)} \wedge u_t^{(2)} \wedge x^{(1)}$ is expressed as

$$w^{(1)} \wedge u_t^{(2)} \wedge x^{(1)} = \sum_{t', y} \beta(t', y) u_{t'}^{(2)} \wedge y, \quad ,$$

where $t' \in \mathbb{Z}$, $y = u_k^{(1)}$ ($k \in \mathbb{Z}^r$) and $\beta(t', y) \in \mathbb{Q}(q)$. Then,

$$w^{(1)} \wedge u_{t+nm}^{(2)} \wedge x^{(1)} = \sum_{t', y} \beta(t', y) u_{t'+nm}^{(2)} \wedge y. \quad .$$

(proof of Claim 1.) We prove it by induction on the size of the partition λ .

If $|\lambda| = 0$, we have $w^{(1)} = u_{[s_1, t+nm+1]}^{(1)}$. Thus, from Lemma 5.5,

$$\begin{aligned} w^{(1)} \wedge u_t^{(2)} \wedge x^{(1)} &= q^{-c+m} u_{[s_1, t+nm+1]}^{(1)} \wedge x^{(1)} \wedge u_t^{(2)} = q^{-c+m} u_{[s_1, t_0+1]}^{(1)} \wedge u_t^{(2)} \\ &= q^{c'} u_t^{(2)} \wedge u_{[s_1, t_0+1]}^{(1)}, \end{aligned}$$

where $c' = \xi(u_{[s_1, t+nm+1]}^{(1)}, u_t^{(2)})$. On the other hand, from $\xi(u_{[s_1, t+nm+1]}^{(1)}, u_{t+nm}^{(2)}) = c'$ and Lemma 5.5,

$$w^{(1)} \wedge u_{t+nm}^{(2)} \wedge x^{(1)} = u_{[s_1, t+nm+1]}^{(1)} \wedge u_{t+nm}^{(2)} \wedge x^{(1)} = q^{c'} u_{t+nm}^{(2)} \wedge u_{[s_1, t_0+1]}^{(1)}.$$

Suppose that $|\lambda| > 0$. Let $\check{w}^{(1)}$ be the simple finite q -wedge removing the first component from $w^{(1)}$, i.e.

$$\check{w}^{(1)} = u_{\lambda_2+s_1-1}^{(1)} \wedge u_{\lambda_3+s_1-2}^{(1)} \wedge \cdots \wedge u_{\lambda_{r-nm-t+t_0}+s_1-(r-nm-t+t_0-1)}^{(1)}.$$

Note that $|\check{w}^{(1)}|_{s_1-1} = |w^{(1)}|_{s_1-1} - \lambda_1 = |\lambda| - \lambda_1$. Since $|\lambda| + |u_t^{(2)}|_{s_2} \leq s_1 - s_2 - nm$ and $\lambda_1 \geq 1$,

$$|\check{w}^{(1)}|_{s_1-1} + |u_t^{(2)}|_{s_2} = |\lambda| - \lambda_1 + |u_t^{(2)}|_{s_2} \leq (s_1 - 1) - s_2 - nm. \quad (31)$$

Thus, the pair $\check{\lambda} = (\lambda_2, \lambda_3, \dots), s_1 - 1, s_2, t, t_0\}$ satisfy the condition of the claim 1. Hence, from the induction hypothesis, if $\check{w}^{(1)} \wedge u_t^{(2)} \wedge x^{(1)}$ is expressed as $\check{w}^{(1)} \wedge u_t^{(2)} \wedge x^{(1)} = \sum_{t', y} \check{\beta}(t', y) u_{t'}^{(2)} \wedge y$ then,

$$\check{w}^{(1)} \wedge u_{t+nm}^{(2)} \wedge x^{(1)} = \sum_{t', y} \check{\beta}(t', y) u_{t'+nm}^{(2)} \wedge y. \quad (32)$$

From Corollary 5.2 (i) and (ii), for all $t', y = u_k^{(1)} = u_{k_2}^{(1)} \wedge \cdots \wedge u_{k_r}^{(1)}$ such that $\check{\beta}(t', y) \neq 0$,

$$|\check{w}^{(1)}|_{s_1-1} + |u_t^{(2)}|_{s_2} = |y|_{s_1-1} + |u_{t'}^{(2)}|_{s_2} \quad \text{and} \quad \lambda_2 + s_1 - 1 \geq t' + nm \geq t + nm. \quad (33)$$

Let $a = \lambda_1 + s_1$. From Corollary 5.2 (ii), for $t' \leq a - nm$

$$u_a^{(1)} \wedge u_{t'}^{(2)} = \sum_{j=0}^{a-t'} C_j(q) u_{t'+j}^{(2)} \wedge u_{a-j}^{(1)}, \quad (34)$$

$$u_a^{(1)} \wedge u_{t'+nm}^{(2)} = \sum_{j=0}^{a-t'-nm} C'_j(q) u_{t'+nm+j}^{(2)} \wedge u_{a-j}^{(1)}, \quad (35)$$

where $C_j(q) \in \mathbb{Q}(q)$. Note that if $0 \leq j \leq a - t' - nm$ then $C_j(q) = C'_j(q)$ because of Corollary 5.3. From (32), (34) and (35), we have

$$\begin{aligned} w^{(1)} \wedge u_t^{(2)} \wedge x^{(1)} &= \sum_{t', y} \sum_{j=0}^{a-t'} C_j(q) \check{\beta}(t', y) u_{t'+j}^{(2)} \wedge u_{a-j}^{(1)} \wedge y, \\ w^{(1)} \wedge u_{t+nm}^{(2)} \wedge x^{(1)} &= \sum_{t', y} \sum_{j=0}^{a-t'-nm} C_j(q) \check{\beta}(t', y) u_{t'+nm+j}^{(2)} \wedge u_{a-j}^{(1)} \wedge y. \end{aligned}$$

Therefore the following claim proves Claim 1. and Lemma 5.12.

Claim 2. Let $a - t' - nm + 1 \leq j \leq a - t'$. Then $u_{a-j}^{(1)} \wedge y = 0$.

(proof of Claim 2.) Let $\gamma = a - t' - j$. Then, $t' + \gamma = a - j$ and $0 \leq \gamma < nm$. We exchange j for γ . i.e. we prove that $u_{t'+\gamma}^{(1)} \wedge y = 0$ if $0 \leq \gamma < nm$. To prove it, we use Lemma 5.9.

Note that the length of the finite q -wedge $u_{t'+\gamma}^{(1)} \wedge y$ is equal to that of $v^{(1)}$, that is $r = s_1 - t_0$. From Corollary 5.2 (ii), it is clear that $\min\{t' + \gamma, k_2, k_3, \dots, k_r\} \geq t_0 + 1 = s_1 - (r - 1)$, where $y = u_k^{(1)} = u_{k_2}^{(1)} \wedge \dots \wedge u_{k_r}^{(1)}$. Thus, it is enough that we prove the pair $(y, u_{t'}^{(2)}, s_1 - 1, s_2)$ is nm -dominant. From (31) and (33),

$$|y|_{s_1-1} + |u_{t'}^{(2)}|_{s_2} = |\check{w}^{(1)}|_{s_1-1} + |u_{t'}^{(2)}|_{s_2} \leq s_1 - s_2 - nm.$$

□

□

□

5.4 Bar involution and dominant finite q -wedges

Definition 5.14. Let $v = u_k = u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r}$ be a finite q -wedge. We define a finite q -wedge \overleftarrow{v} as

$$\overleftarrow{v} = u_{k_r} \wedge u_{k_{r-1}} \wedge \dots \wedge u_{k_2} \wedge u_{k_1}.$$

Definition 5.15. Let $1 \leq d_1 < d_2 < \dots < d_b < j < d_{b+1} < \dots < d_c \leq \ell$ and $v^{(d_i)} = u_{k_1}^{(i)} \wedge u_{k_2}^{(i)} \wedge \dots$ ($1 \leq i \leq c$) be simple finite q -wedges. We define the action of $B'_{-m}[j]$ on the finite q -wedge $v = v^{(d_1)} \wedge v^{(d_2)} \wedge \dots \wedge v^{(d_b)} \wedge v^{(j)} \wedge v^{(d_{b+1})} \wedge \dots \wedge v^{(d_c)}$ as

$$B'_{-m}[j] v = q^{-bm} v^{(d_1)} \wedge v^{(d_2)} \wedge \dots \wedge v^{(d_b)} \wedge (B_{-m} v^{(j)}) \wedge v^{(d_{b+1})} \wedge \dots \wedge v^{(d_c)},$$

where $B_{-m} v^{(j)}$ means the level one action of B_{-m} on $v^{(j)}$. i.e. if $v^{(j)} = u_{k_1}^{(j)} \wedge u_{k_2}^{(j)} \wedge \dots \wedge u_{k_r}^{(j)}$, then

$$B_{-m} v^{(j)} = \sum_{i=1}^r u_{k_1}^{(j)} \wedge \dots \wedge u_{k_{i+nm}}^{(j)} \wedge \dots \wedge u_{k_r}^{(j)}.$$

We also define $B_{-m}[j, \ell]$ on finite q -wedges as similarly in Definition 4.4. i.e.

$$B_{-m}[j, \ell] = \sum_{i=j}^{\ell} q^{(i-j)m} B'_{-m}[i]. \quad (36)$$

Remark. From [Ugl00, Proposition 5.3.(i)], if $|\lambda; s\rangle$ is nm -dominant, then the above definition of $B'_{-m}[j]$ and $B_{-m}[j, \ell]$ on finite q -wedges coincides with that on $|\lambda; s\rangle$.

Lemma 5.16. Let $m \geq 0$ and $1 \leq d_1 < \dots < d_b < j$. Let t, t_0 be two integers such that $t \geq t_0$. Let $\lambda^{(d_1)}, \lambda^{(d_2)}, \dots, \lambda^{(d_b)} \in \Pi$ such that $|\lambda^{(d_1)}| + \dots + |\lambda^{(d_b)}| + |u_t^{(j)}|_{s_j} \leq s_{d_i} - s_j - nm$ for all $1 \leq i \leq b$. For $1 \leq i \leq b$, put $r_i = s_{d_i} - t_0$ and

$$v^{(d_i)} = v_{\lambda^{(d_i)}, r_i}^{(d_i)} = u_{\lambda_1^{(d_i)} + s_{d_i}}^{(d_i)} \wedge u_{\lambda_2^{(d_i)} + s_{d_i} - 1}^{(d_i)} \wedge \dots \wedge u_{\lambda_{r_i}^{(d_i)} + s_{d_i} - r_i + 1}^{(d_i)}.$$

Set $v = v^{(d_1)} \wedge v^{(d_2)} \wedge \dots \wedge v^{(d_b)}$. If $v \wedge u_t^{(j)}$ is expressed as

$$v \wedge u_t^{(j)} = \sum_{t', y} \alpha(t', y) u_{t'}^{(j)} \wedge y, \quad (37)$$

where $t' \in \mathbb{Z}$, $y = y^{(d_1)} \wedge \dots \wedge y^{(d_b)}$ (for all $1 \leq i \leq b$, $y^{(d_i)} = u_{\mathbf{k}^{(d_i)}}^{(d_i)}$ ($\mathbf{k}^{(i)} \in \mathbb{Z}^{r_i}$)) and $\alpha(t', y) \in \mathbb{Q}(q)$. Then,

(i) For all $1 \leq i \leq b$, $|v^{(d_i)}|_{s_i} = |\lambda^{(d_i)}|$.

(ii) For all t' and y , if $\alpha(t', y) \neq 0$, then $|y^{(d_1)}| + \dots + |y^{(d_b)}| + |u_{t'}^{(j)}|_{s_j} \leq s_{d_i} - s_j - nm$.

(iii) $v \wedge u_{t+nm}^{(j)} = q^{2bm} \sum_{t', y} \alpha(t', y) u_{t'+nm}^{(j)} \wedge y$.

(iv) $u_t^{(j)} \wedge \overleftarrow{v} = \sum_{t', y} \overline{\alpha(t', y)} \overleftarrow{y} \wedge u_{t'}^{(j)}$, $u_{t+nm}^{(j)} \wedge \overleftarrow{v} = q^{-2bm} \sum_{t', y} \overline{\alpha(t', y)} \overleftarrow{y} \wedge u_{t'+nm}^{(j)}$.

(v) $\overline{B'_{-m}[j]}(v \wedge u_t^{(j)}) = B'_{-m}[j] \overline{v \wedge u_t^{(j)}}$.

Proof. For simplicity, we assume that $d_1 = 1, d_2 = 2, \dots, d_b = b$.

For all $1 \leq i \leq b$, we have

$$|\lambda^{(i)}| \leq |\lambda^{(1)}| + \dots + |\lambda^{(b)}| \leq s_i - s_j - nm - |u_t^{(j)}|_{s_j} = s_i - s_j - nm - t + s_j \leq s_i - t_0 - nm = r_i - nm.$$

This proves (i).

From Corollary 5.2 (i) (see also the remark after Definition 5.6) and (i), we have

$$|y^{(1)}| + \dots + |y^{(b)}| + |u_{t'}^{(j)}|_{s_j} = |v^{(1)}| + \dots + |v^{(b)}| + |u_{t'}^{(j)}|_{s_j} = |\lambda^{(1)}| + \dots + |\lambda^{(b)}| + |u_t^{(j)}|_{s_j} \leq s_i - s_j - nm$$

for all $1 \leq i \leq b$. This proves (ii).

From (ii), we can apply Lemma 5.12 repeatedly and prove (iii).

Put $v \wedge u_t^{(j)} = A(q) u_t^{(j)} \wedge \overleftarrow{v}$. Then, by the definition of bar involution (Definition 2.8) and Corollary 5.2 (iii),

$$\overline{v \wedge u_{t+nm}^{(j)}} = A(q) u_{t+nm}^{(j)} \wedge \overleftarrow{v}, \quad \overline{y \wedge u_{t'}^{(j)}} = A(q) u_{t'}^{(j)} \wedge \overleftarrow{y} \quad \text{and} \quad \overline{y \wedge u_{t'+nm}^{(j)}} = A(q) u_{t'+nm}^{(j)} \wedge \overleftarrow{y}$$

if $\alpha(t', y) \neq 0$. Thus, we obtain (iv) by taking bar involution of (37) and (iii).

Finally, we prove (v). Put $\overline{v \wedge u_t^{(j)}} = A(q) u_t^{(j)} \wedge \overleftarrow{v}$.

$$\begin{aligned}
\overline{B'_{-m}[j] v \wedge u_t^{(j)}} &= B'_{-m}[j] \left(A(q) u_t^{(j)} \wedge \overleftarrow{v} \right) \\
&= A(q) B'_{-m}[j] \left(\sum_{t',y} \overline{\alpha(t',y)} \overleftarrow{y} \wedge u_{t'}^{(j)} \right) \quad (\text{By (iv)}) \\
&= A(q) q^{-bm} \sum_{t',y} \overline{\alpha(t',y)} \overleftarrow{y} \wedge (B_{-m} u_{t'}^{(j)}) \quad (\text{By Definition 5.15}) \\
&= A(q) q^{-bm} \sum_{t',y} \overline{\alpha(t',y)} \overleftarrow{y} \wedge u_{t'+nm}^{(j)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\overline{B'_{-m}[j] (v \wedge u_t^{(j)})} &= \overline{q^{-bm} v \wedge (B_{-m} u_t^{(j)})} = \overline{q^{-bm} v \wedge u_{t+nm}^{(j)}} \quad (\text{By Definition 5.15}) \\
&= q^{bm} A(q) u_{t+nm}^{(j)} \wedge \overleftarrow{v} \\
&= q^{bm} A(q) q^{-2bm} \left(\sum_{t',y} \overline{\alpha(t',y)} \overleftarrow{y} \wedge u_{t'+nm}^{(j)} \right) \quad (\text{By (iv)}) \\
&= A(q) q^{-bm} \sum_{t',y} \overline{\alpha(t',y)} \overleftarrow{y} \wedge u_{t'+nm}^{(j)}.
\end{aligned}$$

□

Corollary 5.17. *Let $m \geq 0$ and $1 \leq d_1 < \dots < d_b < j$. Let N be a positive integer and t_0 be an integer such that $s_j - N \geq t_0$. Let $\lambda^{(d_1)}, \lambda^{(d_2)}, \dots, \lambda^{(d_b)}, \lambda^{(j)} \in \Pi$ such that $|\lambda^{(d_1)}| + \dots + |\lambda^{(d_b)}| + |\lambda^{(j)}| \leq s_{d_i} - s_j - nm$ for all $1 \leq i \leq b$. Put*

$$w^{(j)} = v_{\lambda^{(j)}, N}^{(j)} = u_{\lambda_1^{(j)} + s_j}^{(j)} \wedge u_{\lambda_2^{(j)} + s_j - 1}^{(j)} \wedge \dots \wedge u_{\lambda_N^{(j)} + s_j - N + 1}^{(j)}.$$

For $1 \leq i \leq b$, put $r_i = s_{d_i} - t_0$ and

$$v^{(d_i)} = v_{\lambda^{(d_i)}, r_i}^{(d_i)} = u_{\lambda_1^{(d_i)} + s_{d_i}}^{(d_i)} \wedge u_{\lambda_2^{(d_i)} + s_{d_i} - 1}^{(d_i)} \wedge \dots \wedge u_{\lambda_{r_i}^{(d_i)} + s_{d_i} - r_i + 1}^{(d_i)}.$$

Set $v = v^{(d_1)} \wedge v^{(d_2)} \wedge \dots \wedge v^{(d_b)}$. Then,

$$\overline{B'_{-m}[j] (v \wedge w^{(j)})} = B'_{-m}[j] \overline{v \wedge w^{(j)}}.$$

Proof. We prove the assertion by induction on N . If $N = 1$, then the assertion follows from Lemma 5.16 (v).

Suppose that $N > 1$. For $1 \leq i \leq N$, put $t_i = \lambda^{(j)} + s_j - i + 1$ and $\tilde{w}^{(j)} = v_{\lambda^{(j)}, N-1}^{(j)} = u_{t_1}^{(j)} \wedge u_{t_2}^{(j)} \wedge \cdots \wedge u_{t_{N-1}}^{(j)}$. Then, $t_i > t_0$ for any $1 \leq i \leq N$ and $|\tilde{w}^{(j)}|_{s_j} = \lambda_1^{(j)} + \cdots + \lambda_{N-1}^{(j)} \leq |\lambda^{(j)}|$. By induction hypothesis,

$$\overline{B'_{-m}[j](v \wedge \tilde{w}^{(j)})} = B'_{-m}[j] \overline{v \wedge \tilde{w}^{(j)}}. \quad (38)$$

Put $\overline{v \wedge \tilde{w}^{(j)} \wedge u_{t_N}^{(j)}} = A(q) u_{t_N}^{(j)} \wedge \overline{v \wedge \tilde{w}^{(j)}}$ and $\overline{v \wedge \tilde{w}^{(j)}} = C(q) \tilde{w}^{(j)} \wedge \overleftarrow{v}$. Set the expansion of $v \wedge \tilde{w}^{(j)}$ as

$$v \wedge \tilde{w}^{(j)} = \sum_{x, y^{(j)}} \alpha(x, y^{(j)}) y^{(j)} \wedge x, \quad (39)$$

where $x = x^{(d_1)} \wedge \cdots \wedge x^{(d_b)}$ (for all $1 \leq i \leq b$, $x^{(d_i)} = u_{\mathbf{k}^{(d_i)}}^{(d_i)} (\mathbf{k}^{(i)} \in \mathbb{Z}^{r_i})$), $y^{(j)} = u_{\mathbf{k}}^{(j)} (\mathbf{k} \in \mathbb{Z}^{N-1})$ and $\alpha(x, y^{(j)}) \in \mathbb{Q}(q)$. Then,

$$\overline{v \wedge \tilde{w}^{(j)}} = C(q) \sum_{x, y^{(j)}} \overline{\alpha(x, y^{(j)})} \overleftarrow{x} \wedge y^{(j)}. \quad (40)$$

Also set the expansion of $x \wedge u_{t_N}^{(j)}$ as

$$x \wedge u_{t_N}^{(j)} = \sum_{t', z} \beta(t', z) u_{t'}^{(j)} \wedge z, \quad (41)$$

where $z = z^{(d_1)} \wedge \cdots \wedge z^{(d_b)}$ (for all $1 \leq i \leq b$, $z^{(d_i)} = u_{\mathbf{g}^{(d_i)}}^{(d_i)} (\mathbf{g}^{(i)} \in \mathbb{Z}^{r_i})$) and $\beta(t', z) \in \mathbb{Q}(q)$. Then, from Lemma 5.16 (iv)

$$u_{t_N}^{(j)} \wedge \overleftarrow{x} = \sum_{t', z} \overline{\beta(t', z)} \overleftarrow{z} \wedge u_{t'}^{(j)}, \quad u_{t_N+nm}^{(j)} \wedge \overleftarrow{x} = q^{-2bm} \sum_{t', z} \overline{\beta(t', z)} \overleftarrow{z} \wedge u_{t'+nm}^{(j)}. \quad (42)$$

Under the above preparation,

$$\begin{aligned}
& \overline{B'_{-m}[j] (v \wedge w^{(j)})} \\
&= \overline{B'_{-m}[j] (v \wedge \tilde{w}^{(j)} \wedge u_{t_N}^{(j)})} \\
&= \overline{q^{-bm} v \wedge (B_{-m} \tilde{w}^{(j)}) \wedge u_{t_N}^{(j)}} + \overline{q^{-bm} v \wedge \tilde{w}^{(j)} \wedge u_{t_N+nm}^{(j)}} \\
&= q^{bm} A(q) u_{t_N}^{(j)} \wedge \overline{v \wedge (B_{-m} \tilde{w}^{(j)})} + q^{bm} A(q) u_{t_N+nm}^{(j)} \wedge \overline{v \wedge \tilde{w}^{(j)}} \\
&= A(q) u_{t_N}^{(j)} \wedge (B'_{-m}[j] \overline{v \wedge \tilde{w}^{(j)}}) + q^{bm} A(q) u_{t_N+nm}^{(j)} \wedge \overline{v \wedge \tilde{w}^{(j)}} \quad (\text{By (38)}) \\
&= A(q) C(q) u_{t_N}^{(j)} \wedge B'_{-m}[j] \left(\sum_{x, y^{(j)}} \overline{\alpha(x, y^{(j)})} \overleftarrow{x} \wedge y^{(j)} \right) + q^{bm} A(q) C(q) u_{t_N+nm}^{(j)} \wedge \left(\sum_{x, y^{(j)}} \overline{\alpha(x, y^{(j)})} \overleftarrow{x} \wedge y^{(j)} \right) \\
&\quad (\text{By (40)}) \\
&= q^{-bm} A(q) C(q) \sum_{x, y^{(j)}} \overline{\alpha(x, y^{(j)})} u_{t_N}^{(j)} \wedge \overleftarrow{x} \wedge (B_{-m} y^{(j)}) + q^{bm} A(q) C(q) \sum_{x, y^{(j)}} \overline{\alpha(x, y^{(j)})} u_{t_N+nm}^{(j)} \wedge \overleftarrow{x} \wedge y^{(j)} \\
&= q^{-bm} A(q) C(q) \sum_{x, y^{(j)}} \sum_{t', z} \overline{\alpha(x, y^{(j)}) \beta(t', z)} \overleftarrow{z} \wedge u_{t'}^{(j)} \wedge (B_{-m} y^{(j)}) \\
&\quad + q^{-bm} A(q) C(q) \sum_{x, y^{(j)}} \sum_{t', z} \overline{\alpha(x, y^{(j)}) \beta(t', z)} \overleftarrow{z} \wedge u_{t'+nm}^{(j)} \wedge y^{(j)} \quad (\text{By (42)}) \\
&= A(q) C(q) \sum_{x, y^{(j)}} \sum_{t', z} \overline{\alpha(x, y^{(j)}) \beta(t', z)} B'_{-m}[j] \left(\overleftarrow{z} \wedge u_{t'}^{(j)} \wedge y^{(j)} \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \overline{B'_{-m}[j] v \wedge w^{(j)}} \\
&= \overline{B'_{-m}[j] v \wedge \tilde{w}^{(j)} \wedge u_{t_N}^{(j)}} \\
&= \overline{B'_{-m}[j] (A(q) u_{t_N}^{(j)} \wedge v \wedge \tilde{w}^{(j)})} \\
&= A(q) C(q) B'_{-m}[j] \left(\sum_{x, y^{(j)}} \overline{\alpha(x, y^{(j)})} u_{t_N}^{(j)} \wedge \overleftarrow{x} \wedge y^{(j)} \right) \quad (\text{By (40)}) \\
&= A(q) C(q) \sum_{x, y^{(j)}} \sum_{t', z} \overline{\alpha(x, y^{(j)}) \beta(t', z)} B'_{-m}[j] \left(\overleftarrow{z} \wedge u_{t'}^{(j)} \wedge y^{(j)} \right) \quad (\text{By (42)}).
\end{aligned}$$

□

Corollary 5.18. *Let $m \geq 0$, $1 \leq j \leq \ell$, $t_0 \in \mathbb{Z}$ and $\lambda \in \Pi^\ell$. Suppose that $|\lambda; s\rangle$ is nm -dominant and $t_0 \leq s_\ell - l(\lambda^{(\ell)})$. For $1 \leq i \leq \ell$, put $r_i = s_i - t_0$ and*

$$v^{(i)} = v_{\lambda^{(i)}, r_i}^{(i)} = u_{\lambda_1^{(i)} + s_i}^{(i)} \wedge u_{\lambda_2^{(i)} + s_i - 1}^{(i)} \wedge \cdots \wedge u_{\lambda_{r_i}^{(i)} + s_i - r_i + 1}^{(i)}.$$

Set $v = v^{(1)} \wedge v^{(2)} \wedge \cdots v^{(j-1)}$ and $w = v^{(j)} \wedge v^{(j+1)} \wedge \cdots v^{(\ell)}$. Then,

$$\overline{B_{-m}[j, \ell] (v \wedge w)} = B_{-m}[j, \ell] \overline{v \wedge w}.$$

Proof. By applying Corollary 5.17 repeatedly, we obtain the assertion. \square

Proof of Proposition 4.6. The statement (ii) follows from Corollary 5.18. We prove (i). For convenience, we define $B_{-m}[\ell + 1, \ell] = B_{-m}[\ell + 2, \ell] = 0$ and $B'_{-m}[\ell + 1] = 0$.

From $B_{-m}[j, \ell] = \sum_{i=j}^{\ell} q^{(i-j)m} B'_{-m}[i]$, we have

$$\begin{aligned} B'_{-m}[j] &= B_{-m}[j, \ell] - \sum_{i=j+1}^{\ell} q^{(i-j)m} B'_{-m}[i] \\ &= B_{-m}[j, \ell] - q^m \sum_{i=j+1}^{\ell} q^{(i-j-1)m} B'_{-m}[i] \\ &= B_{-m}[j, \ell] - q^m B_{-m}[j+1, \ell] \quad . \end{aligned}$$

Thus,

$$\begin{aligned} B_{-m}[j] &= B'_{-m}[j] - q^{-m} B'_{-m}[j+1] \\ &= B_{-m}[j, \ell] - q^m B_{-m}[j+1, \ell] - q^{-m} B_{-m}[j+1, \ell] + B_{-m}[j+2, \ell] \\ &= B_{-m}[j, \ell] - (q^m + q^{-m}) B_{-m}[j+1, \ell] + B_{-m}[j+2, \ell] \quad . \end{aligned}$$

Hence from (ii),

$$\begin{aligned} \overline{B_{-m}[j]} u &= \overline{B_{-m}[j, \ell]} u - (q^m + q^{-m}) \overline{B_{-m}[j+1, \ell]} u + \overline{B_{-m}[j+2, \ell]} u \\ &= B_{-m}[j, \ell] \bar{u} - (q^m + q^{-m}) B_{-m}[j+1, \ell] \bar{u} + B_{-m}[j+2, \ell] \bar{u} = B_{-m}[j] \bar{u} \quad . \end{aligned}$$

\square

References

- [Ari] S. Ariki, *Graded q -Schur algebras*, mathArXiv 0903.3453.
- [GGOR03] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier, *On the category \mathcal{O} for rational Cherednik algebras*, Invent. Math. **154** (2003), no. 3, 617–651, DOI 10.1007/s00222-003-0313-8. MR2018786 (2005f:20010)
- [Iij12] K. Iijima, *A comparison of q -decomposition numbers in the q -deformed Fock spaces of higher levels*, J. Algebra **351** (2012), 426–447, DOI 10.1016/j.jalgebra.2011.09.035. MR2862217
- [JMMO91] M. Jimbo, K. C. Misra, T. Miwa, and M. Okado, *Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}(n)})$ at $q = 0$* , Comm. Math. Phys. **136** (1991), no. 3, 543–566. MR1099695 (93a:17015)

- [LT00] B. Leclerc and J.-Y. Thibon, *Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials*, Combinatorial methods in representation theory (Kyoto, 1998), Adv. Stud. Pure Math., vol. 28, Kinokuniya, Tokyo, 2000, pp. 155–220. MR1864481 (2002k:20014)
- [Lus89] G. Lusztig, *Modular representations and quantum groups*, Classical groups and related topics (Beijing, 1987), Contemp. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1989, pp. 59–77. MR982278 (90a:16008)
- [Rou08] R. Rouquier, *q -Schur algebras and complex reflection groups*, Mosc. Math. J. **8** (2008), no. 1, 119–158, 184. MR2422270 (2010b:20081)
- [Sha] P. Shan, *Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras*, math.arXiv:0811.4549.
- [SW] C. Stroppel and B. Webster, *Quiver Schur algebras and q -Fock space*, mathArXiv 1110.1115.
- [VV99] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. **100** (1999), no. 2, 267–297, DOI 10.1215/S0012-7094-99-10010-X. MR1722955 (2001c:17029)
- [Ugl00] D. Uglov, *Canonical bases of higher-level q -deformed Fock spaces and Kazhdan-Lusztig polynomials*, Physical combinatorics (Kyoto, 1999), Progr. Math., vol. 191, Birkhäuser Boston, Boston, MA, 2000, pp. 249–299. MR1768086 (2001k:17030)
- [Yvo06] X. Yvonne, *A conjecture for q -decomposition matrices of cyclotomic v -Schur algebras*, J. Algebra **304** (2006), no. 1, 419–456, DOI 10.1016/j.jalgebra.2006.03.048. MR2256400 (2008d:16051)